ON SEMI-GROUPS OF UNBOUNDED NORMAL OPERATORS

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In this note we discuss the integral representation of a semi-group of unbounded normal operators. The result for a semi-group of bounded normal operators can be found in Sz. Nagy [4]. Recently A. Devinatz [1] obtained a similar result for semi-groups of unbounded self-adjoint operators. The following theorem is proved.

THEOREM. *Let* $\{N_t;\ t>0\}$ *be a semi-group (i.e.,* $N_tN_s=N_{t+s}$*) of normal operators on a Hilbert space* $\mathfrak{H}$. *Let* D_t *be the domain of* N_t *(each* D_t *dense in* $\mathfrak{H}$*) and let* $D=\bigcap_{t>0} D_t$ *then we suppose that* N_tx *is weakly continuous as a function of* t $(t>0)$ *for each fixed* $x\in D$. *Then there exists a unique complex spectral resolution* $K(\Lambda)$ *whose support is contained in* $\lambda_1\geqq 0$, $\lambda=\lambda_1+i\lambda_2$, *such that*

$$N_t = \int_{\lambda_1\geqq 0} \lambda_1^t e^{i\lambda_2 t} K(d\lambda), \qquad t>0. \tag{1}$$

PROOF. First recall that if N is a normal operator then $N=AU=UA$ where U is unitary and A is self-adjoint and has the same domain as N. In fact if $K(\Lambda)$ is the spectral resolution of N then we can define A and U as follows,

$$A = \int |\lambda|\, K(d\lambda), \tag{2}$$

$$U = \int s(\lambda)K(d\lambda) \quad \text{where} \quad s(\lambda) = \begin{cases} \lambda/|\lambda|, & \lambda\neq 0, \\ 1, & \lambda = 0. \end{cases} \tag{3}$$

Let us also note that if p is an integer then $(N^p)^*=(A^pU^p)^*=A^pU^{-p}$ and that $(N^*)^p=(AU^*)^p=A^pU^{-p}$, hence $(N^p)^*=(N^*)^p$.

We now prove a series of simple statements which taken together yield our theorem.

(a) If t and s are commensurable then $N_tN_s^*=N_s^*N_t$. This is a trivial consequence of the semi-group property of $\{N_t,\ t>0\}$ and the fact that $(N_t^*)^n=(N_t^n)^*$.

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(b) If $x \in D$ then $N_t N_s^* x = N_s^* N_t x$ for all t, $s > 0$. First if $x \in D$ then $N_t x \in D$ and $N_t^* x \in D$ for all $t > 0$, since if $x \in D$ then $N_{t+s} x$ exists and $N_s N_t x = N_{t+s} x$. Thus $N_t x \in D_s$ for all s and hence is in D. To show $N_t^* x \in D$, let $t_0 > 0$ be fixed, then $N_s x \in D$ for all $s > 0$ as seen above, hence $N_{t_0}^* N_s x$ exists. (Domain of $N_{t_0}^*$ is D_{t_0} since N_{t_0} is normal.) Let $s = nt_0$, $n = 1, 2, \cdots$, then since s and t_0 are commensurable $N_{t_0}^* N_s x = N_s(N_{t_0}^* x)$ or $N_{t_0}^* x \in D_{nt_0}$ for $n = 1, 2 \cdots$. But by the semi-group property of N_t, $t > 0$, the D_t's are a decreasing collection of sets and therefore $D = \bigcap_{t>0} D_t = \lim_{t\to\infty} D_t = \lim_{n\to\infty} D_{nt_0}$ since $t_0 > 0$. Thus $N_{t_0}^* x \in D$.

If t, s are given and $x \in D$ then $N_t N_s^* x$ and $N_s^* N_t x$ both exist. Let $t_n \to t$, t_n and s commensurable. A standard argument making use of the continuity property of $\{N_t,\ t > 0\}$ shows that for a fixed $x \in D$ $(N_s^* N_t x, y) = (N_t N_s^* x, y)$ for all $y \in D_s$, but D_s is dense in $\mathcal{H}$ and hence $N_s^* N_t x = N_t N_s^* x$.

(c) If we define $A_t = N_{t/2}^* N_{t/2}$, $t > 0$, then $\{A_t,\ t > 0\}$ is a semi-group of self-adjoint operators such that

$$A_t = A_1^t = \int_0^\infty \lambda_1^t dE(\lambda_1), \qquad t > 0. \tag{4}$$

It is clear from the definition that each A_t is a positive self-adjoint operator and moreover $A_t^2 = N_t^* N_t$. Since a positive self-adjoint operator has a unique positive self-adjoint square root it follows that A_t is the operator defined by (2) for N_t. Thus $D_{A_t} = D_t$ and $D = \bigcap_{t>0} D_{A_t}$.

If $x \in D$ then $A_t A_s x = N_{t/2}^* N_{t/2} N_{s/2}^* N_{s/2} x$ and this is defined since $N_t D \subset D$ and $N_t^* D \subset D$ for all $t > 0$. Moreover by (b)

$$A_t A_s x = N_{t/2}^* N_{s/2}^* N_{t/2} N_{s/2} x = N_{(t+s)/2}^* N_{(t+s)/2} x = A_{t+s} x,$$

since in general $N_{(t+s)/2}^* \supset N_{t/2}^* N_{s/2}^*$. Also

$$\begin{aligned} A_t A_s x &= N_{s/2}^* N_{t/2}^* N_{s/2} N_{t/2} x \\ &= N_{s/2}^* N_{s/2} N_{t/2}^* N_{t/2} x = A_s A_t x. \end{aligned}$$

Consequently if $x \in D$, $A_t A_s x = A_s A_t x = A_{s+t} x$. Now the same argument as used by Sz. Nagy [4, p. 74] shows that $(A_t x, x)$ for each fixed $x \in D$ is bounded above as a function of t in every interval $0 < a \leqq t \leqq b$. This implies, Sz. Nagy [4, p. 73], that $(A_t x, x)$ is continuous for $t > 0$ and $x \in D$. We would like to apply Devinatz's theorem at this point to the A_t's but we do not know a priori that the A_t's form a semi-group. The following argument is almost word for word that of Devinatz [1, p. 102]. Define $H_t = A_1^t$. Clearly $\{H_t; t > 0\}$

is a semi-group. In addition using (a) we see that $H_n = A_1^n = (N_{1/2}^* N_{1/2})^n = N_{n/2}^* N_{n/2} = A_n$. Furthermore the uniqueness of square roots of positive self-adjoint operators implies that for any integers n, m, $H_{n/2^m} = A_{n/2^m}$. Now, there exists a countable set of mutually orthogonal manifolds $\{M_k\}_1^\infty$, whose direct sum is the whole space and such that, for all $t>0$, $H_t = \sum_{k=1}^\infty \oplus H_t^{(k)}$, where $H_t^{(k)}$ is a bounded self-adjoint operator on M_k and is the restriction of H_t to M_k (Sz. Nagy [**4**, p. 49]). That is, $x \in D_{H_t}$ if and only if $\sum_{k=1}^\infty \|H_t^{(k)} x_k\|^2 < \infty$, where $x_k \in M_k$, and $x = \sum_{k=1}^\infty x_k$. Then $H_t x = \sum_{k=1}^\infty H_t^{(k)} x_k$.

Given any $t>0$ there exists an $m/2^n \geqq t$. From the semi-group property of $\{N_t, t>0\}$ we know that $D_{m/2^n} \subset D_t$ (D_t is also the domain of A_t). Thus for every $x_k \in M_k$, $x_k \in D_{m/2^n} \subset D_t$. Consequently, since $H_{m/2^n} = A_{m/2^n}$ and by the continuity of $(A_t x_k, x_k)$ and $(H_t x_k, x_k)$ as functions of t, we must have $A_t x_k = H_t^{(k)} x_k = H_t x_k$. This implies that $H_t = A_t$ (Sz. Nagy [**4**, p. 35]), and hence (4) is proved.

For any $t>0$ the above argument shows that $M_k \subset D_t$ for $k=1, 2, \cdots$, and hence $M_k \subset D$ for all k. Thus if $x \in \mathcal{H}$ and $x = \sum_{k=1}^\infty x_k$ then $y_n = \sum_{k=1}^n x_k$ is in D and $y_n \to x$. That is D is dense in $\mathcal{H}$. Moreover for any $t>0$ if $x \in D_t$ then $A_t y_n = \sum_{k=1}^n A_t x_k = \sum_{k=1}^n H_t^{(k)} x_k \to H_t x = A_t x$. Thus for any $x \in D_t$ there exists a sequence $y_n \in D$ such that $y_n \to x$ and $A_t y_n \to A_t x$.

For each $t>0$ let U_t be the unitary operator defined by (3) such that $N_t = A_t U_t = U_t A_t$. From (2) and (4) it follows that A_t and N_s have the same null space, $\mathfrak{N}$, for all t, $s>0$. $\mathfrak{N}$ is a closed linear manifold since the operators in question are closed. If we write $\mathcal{H} = \mathfrak{R} \oplus \mathfrak{N}$ where $\mathfrak{R}$ is the orthogonal complement of $\mathfrak{N}$, then $\mathfrak{R}$ can be characterized as either the closure of R_{A_t} or the closure of R_{N_t} for any $t>0$. (If T is an operator R_T denotes the range of T.) Thus $\mathfrak{N}$ is the null space of A_t, N_t, and N_t^* for all $t>0$ and if we write (3) with the proper subscript we see that $\mathfrak{N}$ is also the null space of U_t for all $t>0$. (Note that $K_t(\{0\})$ is the projection on $\mathfrak{N}$.) It is now clear that all of the above operators are reduced by $\mathfrak{N}$. (A normal operator is always reduced by its null space.) Thus we can restrict all the operators in question to $\mathfrak{R}$. We assume $\mathfrak{R} \neq \{0\}$ since in this case everything is trivial.

(d) If $\tilde{D} = D \cap \mathfrak{R}$ then $\tilde{D}$ is dense in $\mathfrak{R}$. Assume $\tilde{D}$ not dense in $\mathfrak{R}$ then there exists $r \in \mathfrak{R}$, $r \neq 0$, such that $r \perp \tilde{D}$. Let $x \in D$, then $x = x_R + n$, $x_R \in \mathfrak{R}$, $n \in \mathfrak{N}$. Since $n \in D$ and D is linear we see that $x_R = x - n \in D$ and hence in $\tilde{D}$. Therefore $(r, x) = (r, x_R) + (r, n) = 0$ which implies that $r = 0$ as an element of $\mathcal{H}$ since D is dense in $\mathcal{H}$. But this contradicts the fact that $r \neq 0$ as an element of $\mathfrak{R}$.

(e) In this section all operators are considered as operators on $\mathfrak{R}$.

If we define $U_0=I$ and $U_{-t}=U_t^{-1}$ then $\{U_t;\ -\infty<t<\infty\}$ is a strongly continuous group of unitary operators on $\mathfrak{R}$.

First we show that $A_tU_sx=U_sA_tx$ for $x\in\tilde{D}$. Note that $x\in\tilde{D}$ implies $A_tx\in\tilde{D}$ by the same argument as that used in (b), and also that (b) and the definition of A_t implies $A_tN_sx=N_sA_tx$ for all $x\in\tilde{D}$. Thus for all $x\in\tilde{D}$ we have

$$A_tN_sx = N_sA_tx, \qquad A_tA_sU_sx = A_sU_sA_tx.$$

However, the semi-group property of $\{A_t, t>0\}$ implies $A_tA_s=A_sA_t$, hence

$$A_sA_tU_sx = A_sU_sA_tx$$

and from this we conclude that $A_tU_sx=U_sA_tx$ since A_s is one-to-one in $\mathfrak{R}$, i.e., A_s^{-1} exists. A consequence of this is that $U_sx\in\tilde{D}$ if $x\in\tilde{D}$. Therefore for $x\in\tilde{D}$, $N_tN_sx=N_{t+s}x=A_{t+s}U_{t+s}x=A_tA_sU_{t+s}x$ or $U_{t+s}x=A_s^{-1}A_t^{-1}A_tU_tA_sU_sx=U_tU_sx$. But the U_t's are bounded and $\tilde{D}$ is dense in $\mathfrak{R}$ thus $U_tU_s=U_{t+s}=U_sU_t$ and if we define $U_0=I$ and $U_{-t}=U_t^*$ it is clear that $\{U_t;\ -\infty<t<\infty\}$ is a group of unitary operators on $\mathfrak{R}$.

We now investigate the continuity properties of this group. To this end we first note that an immediate consequence of (4) is that A_t is strongly continuous on $\tilde{D}$ even at $t=0$, and that A_tx is strongly left continuous at $t=t_0$ for $x\in\tilde{D}_{t_0}=D_{t_0}\cap\mathfrak{R}$ (if $s\leqq t$ $D_t\subset D_s$ and hence $\tilde{D}_t\subset\tilde{D}_s$). Suppose $t_0>0$ and $t_n\uparrow t_0$, $0<t_n<t_0$, then $\tilde{D}_{t_0}\subset\tilde{D}_{t_n}$. Let $x\in R_{A_{t_0}}$, $y\in\tilde{D}$ then $x=A_{t_0}z$ where $z\in\tilde{D}_{t_0}\subset\tilde{D}_{t_n}$. Thus we have

$$\begin{aligned}
|(U_{-t_n}x, y) - (U_{-t_0}x, y)| &= |(U_{-t_n}A_{t_0}z, y) - (U_{-t_0}A_{t_0}z, y)| \\
&\leqq |(U_{-t_n}[A_{t_0} - A_{t_n}]z, y)| + |([U_{-t_n}A_{t_n} - U_{-t_0}A_{t_0}]z, y)| \\
&\leqq \|(A_{t_0} - A_{t_n})z\|\cdot\|y\| + |(N_{t_n}^*z, y) - (N_{t_0}^*z, y)| \\
&= \|(A_{t_0} - A_{t_n})z\|\cdot\|y\| + |(z, N_{t_n}y) - (z, N_{t_0}y)| \\
&\to 0 \text{ as } t_n\uparrow t_0.
\end{aligned}$$

Since $R_{A_{t_0}}$ and $\tilde{D}$ are dense in $\mathfrak{R}$ and $\|U_t\|=1$ it follows that $\{U_t, t<0\}$ is weakly right continuous. However, weak right continuity at any one point t_0 implies weak right continuity for all t since $([U_{t+h}-U_t]x, y)=([U_{t_0+h}-U_{t_0}]x,\ U_{t_0-t}y)$. A minor modification in the proof of Theorem 9.2.2 in Hille [3] shows that U_t is strongly continuous for all t. This completes the proof of (e).

From (3) we see that U_t is the identity on $\mathfrak{N}$ and thus $\{U_t;\ -\infty<t<\infty\}$ is a strongly continuous group of unitary operators on all of $\mathfrak{H}$. (We no longer restrict the operators in question to $\mathfrak{R}$.) The spectral theorem for unitary groups guarantees the existence

of unique spectral resolution $F(\lambda_2)$ such that

$$U_t = \int_{-\infty}^{\infty} e^{i\lambda_2 t} dF(\lambda_2), \qquad -\infty < t < \infty. \tag{5}$$

In (e) we saw that $A_t U_s x = U_s A_t x$ for all $x \in \tilde{D}$. But for any $x \in D$ we have $x = n + r$ where $n \in \mathfrak{N}$ and $r \in \tilde{D}$, hence $A_t U_s x = U_s A_t x$ for all $x \in D$ since $\mathfrak{N}$ is the common null space of these operators. For any $x \in D_1$ there exists $y_n \in D$ such that $y_n \to x$ and $A_1 y_n \to A_1 x$. See (c). Hence

$$\begin{aligned} U_t A_1 x &= U_t \left[\lim_{n\to\infty} A_1 y_n \right] = \lim_{n\to\infty} U_t A_1 y_n \\ &= \lim_{n\to\infty} A_1 U_t y_n. \end{aligned}$$

Moreover $U_t y_n \to U_t x$ and since A_1 is closed we have $U_t A_1 x = A_1 U_t x$. Thus $U_t A_1 \subset A_1 U_t$ for $-\infty < t < \infty$. Therefore by Fuglede's theorem [2] we obtain $E(\lambda_1) U_t = U_t E(\lambda_1)$ for all t and λ_1 and then finally that $E(\lambda_1) F(\lambda_2) = F(\lambda_2) E(\lambda_1)$ for λ_1 and λ_2.

Putting $K(d\lambda) = E(d\lambda_1) F(d\lambda_2)$ we have

$$N_t = A_t U_t = \int_{\lambda_1 \gtrless 0} \lambda_1^t e^{i\lambda_2 t} K(d\lambda), \qquad t > 0. \tag{6}$$

Clearly $K(\Lambda)$ is unique on $\mathfrak{R}$ but since $\mathfrak{N} = K(\{0\})\mathfrak{H}$ it follows that $K(\Lambda)$ is unique on all of $\mathfrak{H}$.

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