A NOTE ON CONVEX MAPPINGS

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In a recent paper [1], R. F. Gabriel proved the following theorem. "If $f(z)$ is of the form $1/z+a_1z+a_2z^2+\cdots$, regular for $0<|z|<1$, and if

$$|\{f(z), z\}| \leqq 2c_0 \qquad \text{for } |z| < 1,$$

where c_0 is the smallest positive root of the equation

$$2x^{1/2} - \tan x^{1/2} = 0,$$

then $f(z)$ is univalent in $0<|z|<1$ and maps the interior of each circle $|z|=r<1$ onto the exterior of a convex region. The constant c_0 is the largest possible one."

It is the purpose of the present note to establish a more general result, which contains Gabriel's theorem as a special case. The proof will be based, partly, on methods developed in a recent paper of Z. Nehari [2].

THEOREM I. *Let*

$$f(z) = 1/z + a_0 + a_1z + a_2z^2 + \cdots \tag{1}$$

be a function which is regular for $0<|z|<1$. *Let*

$$|\{f(z), z\}| \leqq 2q(|z|), \qquad |z| < 1, \tag{2}$$

where $q(x)$ *is a function with the following properties*:

(3) (a) $q(x)$ *is positive and continuous for* $0\leqq x<1$.

(b) *The differential equation*

$$y''(x) + q(x)y(x) = 0 \tag{4}$$

has a solution $y(x)$ *which does not vanish in* $0<x<1$, *and satisfies the boundary conditions*

$$\begin{gathered} y(0) = 0, \\ \lim_{x\to 1} \frac{xy'(x)}{y(x)} \geqq 1/2. \end{gathered} \tag{5}$$

(*Because of our assumptions, this limit always exists.*) *Then* $f(z)$ *maps the disk* $|z|<1$ *onto the complement of a convex domain. This result is*

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sharp in the sense that in all cases in which $q(z)$ *is an analytic function in* $|z|<1$ *for which* $|q(z)| \leqq q(|z|)$, *the constant* 2 *in* (2) *is the largest possible.*

The proof of the theorem will be based on the following two lemmas.

LEMMA I. *Let* $p(x)$ *be positive and continuous in* $0 \leqq x < 1$, *and let* $y(x)$ *and* $w(x)$ *be solutions of the differential equations*

$$
\begin{aligned}
y''(x) + p(x)y(x) &= 0, \\
w''(x) + \lambda p(x)w(x) &= 0, \qquad \lambda > 1,
\end{aligned} \tag{6}
$$

respectively, which are positive in $0<x<1$ *and for which* $y(0)=w(0)=0$ *and* $\lim_{x\to 1} w'(x)/w(x) > -\infty$. *Then*

$$\lim_{x\to 1} \frac{w'(x)}{w(x)} < \lim_{x\to 1} \frac{y'(x)}{y(x)} \cdot \tag{7}$$

PROOF. We first note that both limits exist. Indeed,

$$\left(\frac{w'(x)}{w(x)}\right)' = -\left(\frac{w'(x)}{w(x)}\right)^2 - \lambda p(x) < 0,$$

which shows that $w'(x)/w(x)$ decreases monotonically with increasing x. Since the limit $-\infty$ is excluded, the existence of a finite limit follows. To obtain the corresponding property of $y(x)$, we note that from (6) and the fact that $y(0)=w(0)=0$,

$$w(x)y'(x) - w'(x)y(x) = (\lambda - 1)\int_0^x p(x)y(x)w(x)dx.$$

Hence,

$$\frac{w'(x)}{w(x)} = \frac{y'(x)}{y(x)} - \frac{(\lambda-1)}{w(x)y(x)}\int_0^x p(x)y(x)w(x)dx, \tag{8}$$

and thus

$$\lim_{x\to 1} \frac{w'(x)}{w(x)} \leqq \lim_{x\to 1} \frac{y'(x)}{y(x)} \cdot \tag{9}$$

In view of the monotonicity of $y'(x)/y(x)$, this proves the existence of $\lim_{x\to 1} y'(x)/y(x)$. To prove that (9), moreover, implies the stronger inequality (7), we have to show that $w(x)$ and $y(x)$ remain bounded if $x\to 1$. Since $\lim_{x\to 1} y'(x)/y(x)$ exists, we have $y'(x)/y(x) < M < \infty$ for $0<x_0<x<1$. Hence

$$\log y(x) = \log y(x_0) + \int_{x_0}^{x} \frac{y'(x)}{y(x)}\,dx \leqq \log y(x_0) + M(x - x_0),$$

and a similar inequality for $w(x)$. This proves the lemma.

LEMMA II. *Let $w(x)$ be a function which is continuous, has a continuous derivative, and satisfies the conditions $w(0)=0$, $w'(0)\neq 0$. Then there is a positive number δ, such that, for r, $1-\delta<r<1$, we have*

$$r\int_0^r w'^2(x)dx \geqq r\int_0^r q(x)w^2(x)dx + (1/2 - \epsilon)w^2(r),$$

with equality holding if, and only if, $w(x)=cy(x)$, where $y(x)$ is a solution of (4), *satisfying* (5).

PROOF. Consider the function

$$\phi(x) = w'(x) - \frac{y'(x)}{y(x)}\,w(x).$$

By our assumptions, $\phi(x)$ is defined for all x in the interval $0\leqq x\leqq r$. The integral

$$\int_0^r \phi^2(x)dx = \int_0^r \left[w'(x) - \frac{y'(x)}{y(x)}\,w(x)\right]^2 dx$$

exists and is non-negative. Expanding and integrating by parts, we have

$$0 \leqq \int_0^r w'^2(x)dx - \frac{y'(r)}{y(r)}\,w^2(r) + \int_0^r w^2(x)\,\frac{y''(x)}{y(x)}\,dx.$$

For small ϵ, it follows from (5) that if r is such that $1-\delta<r<1$, $ry'(r)/y(r)>1/2-\epsilon$. Hence the inequality becomes

$$0 \leqq \int_0^r w'^2(x)dx - \frac{w^2(r)}{r}(1/2 - \epsilon) - \int_0^r w^2(x)q(x)dx,$$

or

$$r\int_0^r w'^2(x)dx \geqq w^2(r)(1/2 - \epsilon) + r\int_0^r w^2(x)q(x)dx.$$

Equality will hold if, and only if, $\phi(x)=0$: that is,

$$w'(x) - \frac{y'(x)}{y(x)}\,w(x) = 0,$$

or $cy(x)=w(x)$.

PROOF OF THEOREM I. Let

$$p(z) = 1/2\{f(z), z\}. \tag{10}$$

Then, by the classical theory of differential equations, $f(z)$ may be written in the form

$$f(z) = \frac{u(z)}{v(z)}, \tag{11}$$

where $u(z)$ and $v(z)$ are linearly independent solutions of the differential equation

$$y''(x) + p(x)y(x) = 0, \tag{12}$$

with $p(z)$ regular in $|z| < 1$. Since $f(z)$ has the form (1), $u(z)$ and $v(z)$ may be so chosen that

$$\begin{aligned} &u(0) = 1, \\ &v(0) = 0, \qquad v'(0) = 1. \end{aligned} \tag{13}$$

A necessary and sufficient condition that $f(z)$ map the disk $|z| < 1$ onto the exterior of a convex domain is that

$$1 + \operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \leqq 0 \qquad \text{for } |z| < 1. \tag{14}$$

Since

$$f'(z) = \frac{u'(z)v(z) - u(z)v'(z)}{v^2(z)} = -\frac{1}{v^2(z)},$$

$$f''(z) = 2\frac{v'(z)}{v^3(z)},$$

(14) is equivalent to

$$1 - 2\operatorname{Re}\left\{\frac{zv'(z)}{v(z)}\right\} \leqq 0$$

or

$$\operatorname{Re}\left\{\frac{zv'(z)}{v(z)}\right\} \geqq 1/2. \tag{15}$$

$\operatorname{Re}\{zv'(z)/v(z)\}$ is a harmonic function and takes its minimum on the boundary, so that if (15) holds for $|z| = r$, the same will be true for $|z| < r$. Hence $f(z)$ will map $|z| < 1$ onto the complement of a convex region, if the condition

$$\text{(16)} \qquad \operatorname{Re}\left\{\frac{zv'(z)}{v(z)}\right\} \geqq 1/2 - \epsilon$$

is satisfied for a sequence of circles $|z| = r_\nu$, $r_{\nu+1} > r_\nu$, $r_\nu \to 1$, provided $\epsilon = \epsilon(r_\nu) \to 0$ for $r_\nu \to 1$. Indeed, the maximum principle shows that (16) will be true for $|z| = r_\mu$ and $\epsilon = \epsilon(r_\nu)$ with $\mu < \nu$. If we let $\nu \to \infty$, it follows that (15) is satisfied for all r_μ and thus throughout $|z| < 1$.

Consider the equation

$$v''(z) + p(z)v(z) = 0, \qquad v(0) = 0, \qquad v'(0) = 1.$$

Multiplying through by $\bar{v}(z)dz$ and integrating along the ray $\theta =$ constant from the origin to the point $z = re^{i\theta}$, $1 - \delta < r < 1$, we have

$$0 = \int_0^z v''(z)\bar{v}(z)dz + \int_0^z p(z)\,|\,v(z)\,|^2 dz.$$

Integrating by parts and multiplying through by z, we obtain

$$0 = z\frac{v'(z)}{v(z)}\,|\,v(re^{i\theta})\,|^2 - r\int_0^r |\,v'(\rho e^{i\theta})\,|^2 d\rho$$
$$+ r\int_0^r e^{2i\theta}p(\rho e^{i\theta})\,|\,v(\rho e^{i\theta})\,|^2 d\rho.$$

On taking real parts, and noting, in view of (2) and (10), that

$$\operatorname{Re}\left\{e^{2i\theta}p(\rho e^{i\theta})\right\} \leqq q(\rho),$$

it follows that

$$\text{(17)} \qquad |\,v(re^{i\theta})\,|^2 \operatorname{Re}\left\{\frac{zv'(z)}{v(z)}\right\} \geqq r\int_0^r |\,v'(\rho e^{i\theta})\,|^2 d\rho - r\int_0^r q(\rho)\,|\,v(\rho e^{i\theta})\,|^2 d\rho.$$

For $z = \rho e^{i\theta}$, $v(z)$ is a function of ρ along the ray $\theta =$ constant. If $v(z) = \sigma(z) + i\tau(z)$, both σ and τ satisfy the conditions of Lemma II. Since $|v(z)|^2 = \sigma^2(z) + \tau^2(z)$, $|v'(z)|^2 = \sigma_r^2(z) + \tau_r^2(z)$, we thus have the inequality

$$\text{(18)} \quad r\int_0^r |v'(\rho e^{i\theta})|^2 d\rho \geqq |\,v(re^{i\theta})\,|^2(1/2 - \epsilon) + r\int_0^r q(\rho)\,|\,v(\rho e^{i\theta})\,|^2 d\rho.$$

Applying (18) to (17), we get

$$|\,v(re^{i\theta})\,|^2 \operatorname{Re}\left\{\frac{zv'(z)}{v(z)}\right\} \geqq |\,v(re^{i\theta})\,|^2(1/2 - \epsilon),$$

or

$$\operatorname{Re}\left\{\frac{zv'(z)}{v(z)}\right\} \geqq 1/2 - \epsilon$$

and the proof is complete, since the conditions for (16) are satisfied.

It remains to be shown that the theorem is sharp in those cases in which $q(z)$ is an analytic function of z in $|z|<1$ for which $|q(z)| \leqq q(|z|)$. Since $q(x)$ satisfies (4), it follows from Lemma I that there exists a constant C, $C>1$, such that the equation

$$y''(x) + Cq(x)y(x) = 0$$

has a solution $y(x)$ for which $y(0)=0$ and $\lim_{x\to 1} (y'(x)/y(x))=1/2$. The function $q_1(x)=Cq(x)$ may thus take the place of $q(x)$ in Theorem I. Let now $v(z)$ be the solution of the equation

$$v''(z) + \lambda q_1(z)v(z) = 0, \qquad \lambda > 1,$$

with the initial conditions $v(0)=0$, $v'(0)=1$. It follows from Lemma I that

$$\lim_{x\to 1} \frac{xv'(x)}{v(x)} < 1/2.$$

There will therefore exist points x, such that

$$\frac{xv'(x)}{v(x)} < 1/2, \qquad 0 < x < 1.$$

But, as shown before, this implies that the function $f(z)$, normalized by (1), which is a solution of $\{f(z), z\}=2\lambda q_1(z)$ does not map $|z|<1$ onto the complement of a convex region. This shows that the constant 2 in (2) is indeed the largest possible.

The case treated by Gabriel, [1], corresponds to $q(z)=c_0=$constant. In view of the above, the exact value of c_0 has to be determined by the requirement that the equation $y''(x)+c_0y(x)=0$ has a solution $y(x)$ such that $y(0)=0$, $y(x)\neq 0$ for $0<x<1$, and $y'(1)/y(1)=1/2$. It follows that c_0 is the smallest positive root of $2x^{1/2}-\tan x^{1/2}=0$.

References

1. R. F. Gabriel, *The Schwarzian derivative and convex functions*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 58–66.

2. Z. Nehari, *Some criteria of univalence*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 700–704.

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