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NOTE ON LINEAR FORMS

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1. There has been some interest in solutions to the equation

$$n = a_0x_0 + a_1x_1 + \cdots + a_sx_s \tag{*}$$

where the a_i are fixed positive integers with $\gcd = 1$ and the x_i are non-negative integers. In particular the question of finding the smallest n for which all greater integers have a solution has been investigated to some extent [1; 2]. It seems that the solution for $s=1$ has been known for some time but that the problem in general remains unsolved for $s>1$. In the paper of A. Brauer cited in the bibliography various upper bounds for the smallest n are given and the actual value of the smallest n is determined for the a_i consecutive integers. The main result of this paper is the determination of this smallest n when the a_i are in arithmetical progression.

2. Our investigation then is with the linear form

$$F = a_0x_0 + \cdots + a_sx_s.$$

Throughout this paragraph we assume $2 \leqq a_0$, $\gcd a_i = 1$ and $a_j = a_0 + jd$. Thus the a_i are in arithmetical progression. Then we have the

THEOREM. *F represents all $n \geqq N$ where*

$$N = \left(\left[\frac{a_0 - 2}{s}\right] + 1\right) \cdot a_0 + (d - 1)(a_0 - 1)$$

with non-negative x_i and does not so represent $N-1$.

The proof of this result breaks down into a series of five lemmas.

LEMMA 1. *The only integers represented by F when $x_0 + \cdots + x_s = m$ are ma_0, ma_0+d, $ma_0+2d, \cdots, ma_0+msd$.*

PROOF. F represents ma_0 for $x_0 = m$, other $x_i = 0$. If F represents $ma_0 + kd$ with $\sum_0^s x_i = m$ and $k < ms$ then $x_i > 0$ for some $i < s$. In the representation of $ma_0 + kd$ replace $x_0, \cdots, x_i, x_{i+1}, \cdots, x_s$ by $x_0, \cdots, x_i - 1, x_{i+1} + 1, \cdots, x_s$. Now F represents $ma_0 + (k+1)d$.

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Smallest is clearly ma_0. By induction this gives ma_0+kd for $k\leqq ms$. But largest is ma_0+msd.

DEFINITION. $[x]^+=$ the smallest integer $\geqq x$.

LEMMA 2. *Let* $1\leqq r\leqq d$ *and* $j_r=\max\ (0,\ [(a_0-rs-1)ds]^+)$. *Then all numbers* $\geqq(r+j_rd)a_0$ *which are congruent to* ra_0 *modulo* d *are represented by* F.

PROOF. Every number represented by F has the form $a_0\sum_0^s x_i+d\sum_0^s ix_i$. Hence those with the property of being $\equiv ra_0 \pmod d$ are those with $a_0\sum_0^s x_i\equiv ra_0 \pmod d$. Since $(a_0, d)=1$ this is equivalent to $\sum_0^s x_i\equiv r \pmod d$. Suppose $m=\sum_0^s x_i\equiv r \pmod d$. Then $m=r+jd$. Now the largest number represented by F for this m is clearly $ma_s=(r+jd)(a_0+sd)$. The next larger number represented by F which is $\equiv ra_0 \pmod d$ is $m'a_0$ where $m'=m+d$. Hence the next larger number will be $(m+d)a_0=(r+(j+1)d)a_0$. By Lemma 1 we have with $\sum x_i=m$ all the numbers $ma_0, \cdots, ma_s$ and only those represented by F. Similarly for $\sum x_i=m+d=m'$ we have only $m'a_0, \cdots, m'a_s$ represented by F. The necessary and sufficient condition that there be no numbers which are $\equiv ra_0 \pmod d$ between the largest of $ma_0, \cdots, ma_s$ which is $ma_s=(r+jd)(a_0+sd)$ and the smallest of $m'a_0, \cdots, m'a_s$ which is $m'a_0=(r+(j+1)d)a_0$ is that $(r+jd)(a_0+sd)+d\geqq(r+(j+1)d)a_0$. This holds if and only if $rs+jsd+1\geqq a_0$. If $rs+1\geqq a_0$ then we can take $j=0=j_r$. Otherwise $j\geqq(a_0-rs-1)/ds$ and so the smallest integral value of j, denoted by j_r, is the smallest integer $\geqq(a_0-rs-1)/ds$. Hence $j_r=\max\ (0,\ [(a_0-rs-1)/ds]^+)$. Since the condition is necessary and sufficient we shall miss the number $(r+j_rd)a_0-d$.

By allowing r to run through the numbers $1, 2, \cdots, d$ we shall get d sets of numbers as follows, where $Q_r=(r+j_rd)a_0$.

$$Q_j, Q_j+d, Q_j+2d, \cdots; \qquad j=1, 2, 3, \cdots, d.$$

All of these numbers are represented by F by Lemma 2. Also by Lemma 2 we know that Q_i-d is not represented by F for any $i=1, \cdots, d$. Since $Q_i\equiv ia_0 \pmod d$ we see that any set of d numbers, one from each row, will give a complete set of residues modulo d. Hence no two numbers in the above array are the same. Now if we let $Q=\max\ \{Q_i\}, i=1, \cdots, d$, we see that if $N\geqq Q$ then N is represented by F. In fact for $N\geqq Q-d+1$ we have N represented by F and we do not have $Q-d$ represented by F. Hence our desired smallest n is just $Q-d+1$.

What remains is to show that $Q-d+1=N$ where N is as given in the theorem.

LEMMA 3. *The Q above is equal to $(k+td)a_0$ where*

$$t = \max\ (0,\ [(a_0 - s - 1)/ds]^+)$$

and

$$k = \begin{cases} d \text{ if } t = 0 \\ \text{largest of } 1, \cdots, d \quad \text{such that } [(a_0 - ks - 1)/ds]^+ = t \text{ if } t \neq 0. \end{cases}$$

PROOF. $Q = \max\ \{Q_i\} = \max\ ((1+j_1d)a_0,\ (2+j_2d)a_0, \cdots, (d+j_dd)a_0)$. Remembering that $j_r = \max\ (0,\ [(a_0-rs-1)/ds]^+)$ we see that if $t=0$ then $[(a_0-s-1)/ds]^+ \leqq 0$ and hence $[(a_0-rs-1)/ds]^+ \leqq 0$ so that $j_r = 0$ for all $r = 1, \cdots, d$. Hence $Q = da_0$. On the other hand if $t \neq 0$ then there is a largest k from 1 to d for which $[(a_0-s-1)/ds]^+ = [(a_0-ks-1)/ds]^+$. For the corresponding term of Q we have

$$\begin{aligned}(k + j_kd)a_0 &= (k + [(a_0 - ks - 1)/ds]^+d)a_0 \\ &= (k + [(a_0 - s - 1)/ds]^+d)a_0 = (k + td)a_0.\end{aligned}$$

This term is certainly larger than $(j+[(a_0-js-1)/ds]^+d)a_0$ for $1 \leqq j < k$. Also for $j > k$ we have $[(a_0-js-1)/ds]^+ < [(a_0-ks-1)/ds]^+$ and so $([(a_0-js-1)/ds]^+ + 1)d \leqq [(a_0-ks-1)/ds]^+d = td$. Hence $[(a_0-js-1)/ds]^+d \leqq (t-1)d$ so

$$(j+[(a_0-js-1)/ds]^+d)a_0 \leqq (j+(t-1)d)a_0 = ((j-d)+td)a_0 < (k+td)a_0.$$

This completes the proof of the lemma.

LEMMA 4. *F represents all numbers $n \geqq N$ where $N = (k+td)a_0 - d + 1$, and k, t are as defined in Lemma 3, but F does not represent $N-1$.*

PROOF. This lemma is just a restatement of the remarks preceding Lemma 3 combined with that lemma.

LEMMA 5. *For t, k defined as in Lemma 3 we have $k+td = [(a_0-2)/s] + d$.*

PROOF. (a) $t=0$. By definition of t we see that $[(a_0-s-1)/ds]^+ \leqq 0$ so $s+1 \geqq a_0$ and $a_0 - 2 \leqq s - 1 < s$. Therefore $[(a_0-2)/s] = 0$. Also, by definition of k we have $k = d$. Hence $k+td = d = [(a_0-2)/s] + d$.

(b) $t \neq 0$. Suppose that

$$\frac{a_0 - s - 1}{ds} = I + \frac{j}{ds}, \quad I \text{ an integer, } 0 \leqq j < ds. \tag{1}$$

Then

$$\frac{a_0 - ks - 1}{ds} = I + \frac{j}{ds} - \frac{k-1}{d} \tag{2}$$

and

$$\frac{a_0 - (k+1)s - 1}{ds} = I + \frac{j}{ds} - \frac{k}{d}\cdot \tag{3}$$

Also by definition of k we know

$$\left[\frac{a_0 - s - 1}{ds}\right]^+ = \left[\frac{a_0 - ks - 1}{ds}\right]^+ > \left[\frac{a_0 - (k+1)s - 1}{ds}\right]^+. \tag{4}$$

We now have two cases; first with $j=0$ and secondly with $j\neq 0$.

(i) $j=0$. From (1), (2), and (4) we see that $(k-1)/d<1$ or $k<d+1$. Also from (1), (3), and (4) we have $k\geqq d$. Hence $k=d$. Now $[(a_0-2)/s]=[(a_0-s-1)/s+(1-1/s)]=(a_0-s-1)/s$ since $(a_0-s-1)/s=dI$ which is an integer while $(1-1/s)<1$. Thus $k+td=d+[(a_0-s-1)/ds]^+d=[(a_0-2)/s]+d$.

(ii) $j\neq 0$. From (1)–(4) we deduce that $j/ds-(k-1)/d>0$ and $j/ds-k/d\leqq 0$. Hence $k<j/s+1$ and $k\geqq j/s$. Therefore $k=[j/s]^+$. Now $[(a_0\leqq s-1)/ds]^+=[I+j/ds]^+=I+1=(a_0-s-1)/ds-j/ds+1$. Hence $k+td=k+[(a_0-s-1)/ds]^+d=[j/s]^++(a_0-s-1)/s-j/s+d$. Now either j/s is an integer or not. We suppose first that it is an integer. Then $[j/s]^+=j/s$. Since $I=(a_0-s-1)/ds-j/ds$ we have $(a_0-s-1)/s=dI+j/s$ which is an integer. Thus $[(a_0-2)/s]=[(a_0-s-1)/s+(1-1/s)]=(a_0-s-1)/s$ and $k+td=(a_0-s-1)/s+d=[(a_0-2)/s]+d$. We now suppose that j/s is not an integer but that it is equal to $J+i/s$ where J is an integer and $0<i<s$. In this case $[j/s]^+=J+1=j/s-i/s+1$. Therefore $k+td=[j/s]^++(a_0-s-1)/s-j/s+d=1-i/s+(a_0-s-1)/s+d$. Now $(a_0-s-1)/s=dI+j/s=dI+J+i/s$ so $[(a_0-2)/s]=[(a_0-s-1)/s+(1-1/s)]=[dI+J+(i-1)/s+1]$. Since $1\leqq i\leqq s-1$ we know $0\leqq(i-1/s)<1$. Therefore $[(a_0-2)/s]=dI+J+1=(a_0-s-1)/s-i/s+1=k+td-d$ and so $[(a_0-2)/s]+d=k+td$. This completes the proof of the lemma.

Putting Lemmas 4 and 5 together gives us the proof of the theorem.

3. In §2 we have disposed of the problem of §1 when the a_i are in arithmetic progression. The simplest case not covered seems to be that where the a_i are general but $s=2$. Even this case seems quite difficult however. We state the following result which is a specialization of the case just mentioned.

Given $F=ax_0+(a+1)x_1+(a+z)x_2$, $z>2$, and the x_i to be nonnegative then the smallest N for which all $n\geqq N$ are represented by F but with $N-1$ not so represented is given by

(a) $((a+1)/z)a+(z-3)a$ when $a\equiv z-1 \pmod z$ and $a\geqq z^2-5z+3$,

(b) $[(a+1)/z](a+z)+(z-3)a$

when $a\not\equiv z-1 \pmod z$ and $a\geqq z^2-4z+2$.

We omit the proof of this result as it is rather long.

It is not hard to find the desired N for specific triples of numbers. For instance when a_0, $a_1=a_0+2$, $a_2=a_0+3$ we find the value of N to be $[x/3]\cdot x+2+x$. If the largest of a_0, a_1, a_2 is sufficiently larger than the other two and those two are relatively prime then the N is easily determined also. In fact if $a_0<a_1<a_2$ and $(a_0, a_1)=1$ and $a_2>(a_0-1)(a_1-1)-a_0$ then $N=(a_0-1)(a_1-1)$.

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ON THE INFINITUDE OF PRIMITIVE k-NONDEFICIENTS

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H. N. Shapiro has defined [1] a k-nondeficient as an integer n which satisfies

$$\sigma(n)/n \geqq k \qquad (k \text{ real}) \tag{1}$$

where $\sigma(n)$ is the sum of the divisors of n. Integers n which do not satisfy (1) are called k-deficient. A primitive k-nondeficient is defined as a k-nondeficient, all of whose proper divisors are k-deficient. In the same paper, Shapiro shows that, in order for an infinite number of primitive k-nondeficients to exist, it is necessary that k be of the form

$$\prod_{i=1}^{m} \frac{p_i^{\alpha_i+1}-1}{(p_i-1)p_i^{\alpha_i}} \prod_{i=m+1}^{n} \frac{p_i}{p_i-1} \tag{2}$$

or, written another way,

$$\prod_{i=1}^{m} \frac{\sigma(p_i^{\alpha_i})}{p_i^{\alpha_i}} \prod_{i=m+1}^{n} \frac{p_i}{p_i-1}$$

where $p_1, p_2, p_3, \cdots, p_n$ are distinct primes and $0\leqq m\leqq n$. In this note we show that, for every k of the form (2), an infinite number of

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