CONVERGENCE OF FINITE DIFFERENCE SOLUTIONS TO A SOLUTION OF THE EQUATION OF THE VIBRATING ROD

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1. Introduction. In solving partial differential equations by machine methods, it is customary to replace the partial differential equations by finite difference equations. Questions of stability and convergence of the finite difference solution then arise naturally. In the recent literature the heat equation $U_t = U_{xx}$ has been thoroughly investigated from this point of view in References [1; 2; 3], and the wave equation $U_{tt} = U_{xx}$ has been similarly treated in [4]. In this paper the stability and convergence of two different finite difference approximations to a partial differential equation of the fourth order will be considered.

2. The problem. The equation and boundary conditions governing the transverse vibrations of a homogeneous rod are

$$(1) \quad \frac{\partial^4 U}{\partial x^4} + \frac{\partial^2 U}{\partial t^2} = 0, \quad 0 < x < 1, t > 0,$$

$$U(0, t) = U(1, t) = U_{xx}(0, t) = U_{xx}(1, t) = 0, \quad t \geq 0.$$

The initial conditions are taken to be

$$(2) \quad U(x, 0) = f(x), \quad U_t(x, 0) = 0, \quad 0 < x < 1.$$

The exact solution of this problem is well known and is given by

$$(3) \quad U(x, t) = \sum_{n=1}^{\infty} a_n \sin n\pi x \cos n^2\pi^2 t,$$

where

$$(4) \quad a_n = 2 \int_0^1 f(x) \sin n\pi x dx.$$

The function $f(x)$ is assumed to have continuous first and second derivatives and piecewise continuous third and fourth derivatives.

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The more general initial condition $U_t(x, 0) = g(x)$ will introduce no additional complexities.

3. A forward finite difference equation. The equation (1) may be replaced by the finite difference equation

$$(5) \quad \begin{aligned} V(x, t + \Delta t) - 2V(x, t) + V(x, t - \Delta t) \\ = -r^2[V(x + 2\Delta x, t) - 4V(x + \Delta x, t) \\ + 6V(x, t) - 4V(x - \Delta x, t) + V(x - 2\Delta x, t)], \end{aligned}$$

where r is the mesh ratio defined by

$$(6) \quad \Delta t = r(\Delta x)^2.$$

Equation (5) is a forward finite difference representation which allows us to calculate successively the values of V along the mesh line $t + \Delta t$ if the values of V are known along the mesh lines t and $t - \Delta t$. Let us divide the interval $0 \leq x \leq 1$ into M equal subintervals so that $\Delta x = 1/M$. By separating variables it is easily shown that a solution of (5) for fixed r and M is

$$(7) \quad V(x, t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \cos \left\{ \frac{tM^2}{r} \arccos \left(1 - 8r^2 \sin^4 \frac{n\pi}{2M} \right) \right\},$$

where b_n are arbitrary constants. Each term of (7) satisfies (5) and the same boundary conditions as $U(x, t)$ at the mesh points, namely

$$(8) \quad V(0, t) = V(1, t) = V_{xx}(0, t) = V_{xx}(1, t) = 0, \quad t \geq 0.$$

Collatz [5] has shown that the finite difference representation (5) under the boundary conditions (8) is stable for a fixed value of the mesh ratio $r \leq 1/2$. A finite difference representation is defined to be stable if round off errors introduced at one line of the computation tend to decay as the computation proceeds. Collatz did not however investigate the convergence of the solution (7) of the difference equation to the solution (3) of the differential equation. In this paper we shall show that with proper choice of the constants b_n the solution (7) will converge to the solution (3) for any arbitrary fixed value of $r > 0$ as $M \rightarrow \infty$.

4. An exact solution of the difference equation (5). Following Leutert [4] we define for any positive integer $M > 1$ another positive integer $k(M)$ such that $k(M) < M^{1/5}$ and $\lim_{M \rightarrow \infty} k(M) = \infty$. In (7) we choose $b_n(M) = 0$ for $n > k(M)$ and we choose the remaining $b_n(M)$ so that for $t > 0$,

$$|b_n(M) - a_n| < \epsilon M^{-1/5} \quad \text{uniformly for } n = 1, 2, \dots, k,$$

whenever $M > M_1(\epsilon)$. Then an exact solution of (5) is

$$(9) \quad V_M(x, t) = \sum_{n=1}^{k(M)} b_n(M) \sin n\pi x \cos \left\{ \frac{tM^2}{r} \arccos \left(1 - 8r^2 \sin^4 \frac{n\pi}{2M} \right) \right\}$$

for $0 \leq x \leq 1$, $t \geq 0$. The initial condition at the line $t=0$ becomes

$$(10) \quad V_M(x, 0) = \sum_{n=1}^{k(M)} b_n(M) \sin n\pi x.$$

One possible way to choose the constants b_n in (9) is to take $b_n = a_n$ ($n=1, 2, \dots, k$). This is the choice which minimizes the integral

$$\int_0^1 [f(x) - V_M(x, 0)]^2 dx.$$

5. The convergence theorem. We establish first the following lemma:

LEMMA I. Let $r > 0$, $0 \leq z \leq \pi M^{-4/5}/2$. Then there exists an $M_2(r)$ such that for $M > M_2$ and for every $\epsilon > 0$

$$(11) \quad \left| 4rz^2 - \arccos \left(1 - 8r^2 \sin^4 \frac{n\pi}{2M} \right) \right| < \frac{r\epsilon}{M^3}.$$

PROOF. Choose $M_3(r)$ so that for all admissible z and for all $M > M_3$,

$$(12) \quad 1 - 8r^2 \sin^4 z > 0.$$

Let $f(z) = 4rz^2 - \arccos(1 - 8r^2 \sin^4 z)$. We then have $f(0) = f'(0) = f''(0) = f'''(0) = 0$ and

$$f'''(z) = \frac{16r \sin 2z (1 + 8r^2 \sin^4 z - 10r^2 \sin^2 z - 8r^4 \sin^6 z)}{(1 - 4r^2 \sin^4 z)^{5/2}}.$$

From (12) we have $4r^2 \sin^4 z < 1/2$. It then follows easily that

$$f'''(z) \leq \frac{16r \sin 2z \cdot 2}{(1/2)^{5/2}} = 32 \cdot 2^{5/2} r \sin 2z,$$

and

$$\begin{aligned} f'''(z) &\geq -16r \sin 2z \cdot 2^{5/2} \left(\frac{10r}{2 \cdot 2^{1/2}} + \frac{8r}{16 \cdot 2^{1/2}} \right) \\ &\geq -352r^2 \sin 2z. \end{aligned}$$

For $0 \leq z \leq \pi/4$, $\sin 2z \leq 2z$ so that

$$-704r^2z \leq f'''(z) \leq 256 \cdot 2^{1/2}rz.$$

Upon integrating three times from 0 to z we obtain

$$-\frac{88}{3}r^2z^4 \leq f(z) \leq \frac{32 \cdot 2^{1/2}}{3}rz^4,$$

or

$$-\frac{11}{12}r^2\pi^4M^{-16/5} \leq f(z) \leq \frac{2 \cdot 2^{1/2}}{3}r\pi^4M^{-16/5}.$$

It is clear that there exists an M_2 such that for $M > M_2$ and for every $\epsilon > 0$, $|f(z)| < \epsilon r/M^3$. This establishes the lemma.

We are now prepared to prove the convergence theorem.

THEOREM I. *Under the assumptions*

(a) r, t, x fixed with $r > 0, t > 0, 0 < x < 1$,

(b) $f(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x, a_n = 2 \int_0^1 f(x) \sin n\pi x dx,$

$|f(x)| < P$ on $0 \leq x \leq 1$ and $f(x)$ satisfies the conditions stated in §2;

(c) $|b_n(M) - a_n| < \epsilon M^{-1/5}$ for $1 \leq n \leq k(M)$ and for all $M > M_1(\epsilon)$;

(d) $k(M) < M^{1/5}, \quad \lim_{M \rightarrow \infty} k(M) = \infty;$

it follows that

$$\lim_{M \rightarrow \infty} V_M(x, t) = U(x, t),$$

or

$$\begin{aligned} \lim_{M \rightarrow \infty} \sum_{n=1}^{k(M)} b_n(M) \sin n\pi x \cos \left\{ \frac{tM^2}{r} \arccos \left(1 - 8r^2 \sin^4 \frac{n\pi}{2M} \right) \right\} \\ = \sum_{n=1}^{\infty} a_n \sin n\pi x \cos n^2\pi^2 t. \end{aligned}$$

PROOF. In (11) set $z = n\pi/2M$. Then for $M > M_2$

$$\left| \frac{M^2 t}{r} \arccos \left(1 - 8r^2 \sin^4 \frac{n\pi}{2M} \right) - n^2\pi^2 t \right| < \frac{\epsilon t}{M}$$

and therefore

$$\left| \cos \frac{M^2 t}{r} \arccos \left(1 - 8r^2 \sin^4 \frac{n\pi}{2M} \right) - \cos n^2 \pi^2 t \right| < \frac{\epsilon t}{M}.$$

Let us write

$$\begin{aligned} V_M(x, t) - U(x, t) &= \sum_{n=1}^k (b_n - a_n) \sin n\pi x \cos n^2 \pi^2 t \\ &\quad + \sum_{n=1}^k [(b_n - a_n) + a_n] \sin n\pi x \\ &\quad \cdot \left\{ \cos \frac{M^2 t}{r} \arccos \left(1 - 8r^2 \sin^4 \frac{n\pi}{2M} \right) - \cos n^2 \pi^2 t \right\} \\ &\quad - \sum_{n=k+1}^{\infty} a_n \sin n\pi x \cos n^2 \pi^2 t \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Using the assumptions of the theorem we obtain

$$\begin{aligned} |I_1| &\leq \epsilon \cdot M^{-1/5} \cdot k(M) \leq \epsilon && \text{for } M > M_1, \\ |I_2| &\leq \epsilon \cdot M^{-1/5} \cdot \epsilon t / M \cdot k(M) \leq \epsilon^2 t && \text{for } M > M_2, \\ |I_3| &\leq 2P \cdot \epsilon t / M \cdot k(M) < 2\epsilon P t && \text{for } M > M_2, \\ |I_4| &\leq \epsilon && \text{for } M > M_4, \end{aligned}$$

the last inequality following from the convergence of the series solution (3) for $U(x, t)$ for fixed t and from assumption (d). Hence for $M > \max(M_1, M_2, M_3, M_4)$ we obtain upon adding the inequalities

$$|V_M(x, t) - U(x, t)| \leq \epsilon(2 + \epsilon t + 2Pt).$$

This establishes the convergence theorem.

6. An implicit finite difference approximation. The forward difference approximation given in §3 is stable only if $r \leq 1/2$. This imposes a severe restriction on the size of the mesh length $\Delta t \leq 1/2M^2$. In order to remove this restriction we replace the differential equation by the following implicit difference approximation:

$$(13) \quad \Delta_t^2 U(x, t) + (r^2/4) \Delta_x^4 [U(x, t + \Delta t) + 2U(x, t) + U(x, t - \Delta t)] = 0,$$

where Δ_x^4 and Δ_t^2 denote the fourth central difference with respect to x and the second central difference with respect to t , respectively. This representation leads to a system of $M-1$ equations in $M-1$ unknowns which must be solved at each line in t . To test the stability of this representation we note first that the error $e(x, t)$ introduced

at one line of a computation satisfies the same difference equation (13) as the function $U(x, t)$. If a solution of the error equation is sought in the form $e(x, t) = f(x)g(t)$, the method of separation of variables yields the two following equations for determining f and g :

$$g(t + \Delta t) - 2 \frac{1 + \lambda}{1 - \lambda} g(t) + g(t - \Delta t) = 0,$$

$$(14) \quad f(x + 2\Delta x) - 4f(x + \Delta x) + 2 \left(3 + \frac{2\lambda}{r^2} \right) f(x) - 4f(x - \Delta x) + f(x - 2\Delta x) = 0.$$

A solution of (14) which satisfies the boundary conditions (8) is $f(x) = \sin \alpha x$. By substituting in (14), the relationship between α and λ is found to be

$$\lambda = -4r^2 \sin^4 \alpha \Delta x / 2.$$

The determinantal equation for $g(t) = \xi^t$ is

$$\xi^{2\Delta t} - 2 \frac{1 + \lambda}{1 - \lambda} \xi^{\Delta t} + 1 = 0.$$

Now errors of the form $f(x)g(t)$ will not grow in time if $|g(t)| \leq 1$. The solution of the determinantal equation is

$$\xi^{\Delta t} = \frac{1 - 4r^2 \sin^4 \alpha \Delta x / 2}{1 + 4r^2 \sin^4 \alpha \Delta x / 2},$$

from which it is apparent that $|\xi^{\Delta t}| \leq 1$ for all $r > 0$ and for all real values of α . Thus the finite difference approximation is stable for all values of the mesh ratio r .

By separating variables a solution of (13) which satisfies the same boundary conditions as $U(x, t)$ is found to be

$$(15) \quad \bar{V}_M(x, t) = \sum_{n=1}^{\infty} \bar{b}_n(M) \sin n\pi x \cos \left\{ \frac{M^2 t}{r} \cdot \arccos \frac{1 - 4r^2 \sin^4 (n\pi/2M)}{1 + 4r^2 \sin^4 (n\pi/2M)} \right\}.$$

7. Convergence of the implicit difference solution. We first prove the following lemma which is very similar to Lemma I.

LEMMA II. Let r, t be fixed, $t > 0$, $0 \leq z \leq \pi M^{-4/5}/2$. Then there exists an integer $\bar{M}_1(r)$ such that for all $M > \bar{M}_1$,

$$\left| 4rz^2 - \arccos \left(\frac{1 - 4r^2 \sin^4 z}{1 + 4r^2 \sin^4 z} \right) \right| < \frac{\epsilon r}{M^3}.$$

PROOF. Choose $\overline{M}_3$ so that for $M > \overline{M}_3$

$$1 - 4r^2 \sin^4 z > 0.$$

Let

$$f(z) = 4rz^2 - \arccos \frac{1 - 4r^2 \sin^4 z}{1 + 4r^2 \sin^4 z}.$$

Then just as in Lemma I, $f(0) = f'(0) = f''(0) = f'''(0) = 0$ and

$$f'''(z) = \frac{16r \sin 2z}{(1 + 4r^2 \sin^4 z)^3} (1 + 2r^2 \sin^2 z - 24r^2 \sin^4 z - 48r^4 \sin^6 z + 16r^4 \sin^8 z).$$

Using $r^2 \sin^4 z < 1/4$ we establish the following bounds on $f'''(z)$,

$$-8r(1 + r)z^4 \leq f'''(z) \leq \frac{8r}{3}(1 + 5r)z^4,$$

and therefore,

$$-1/2r(1 + r)\pi^4 M^{-16/5} \leq f(z) \leq \frac{1}{6} r(1 + 5r)\pi^4 M^{-16/5}.$$

Thus for $M > \overline{M}_2$,

$$|f(z)| < \epsilon r / M^3.$$

THEOREM II. Under the same assumptions as in Theorem I,

$$\lim_{M \rightarrow \infty} \overline{V}_M(x, t) = U(x, t).$$

The proof of the convergence theorem now follows in exactly the same manner as in Theorem I.

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