GROUPS ON E^n WITH $(n-2)$ -DIMENSIONAL ORBITS¹

D. MONTGOMERY, H. SAMELSON, AND C. T. YANG

1. Introduction. It is the purpose of this paper to prove the following:

THEOREM A. *Let G be a compact connected Lie group which acts differentiably on $E = E^n$ in such a way that the highest dimension of any orbit is either $(n-1)$ or $(n-2)$. Then in properly chosen coordinates the action of G is linear.*

To say that G acts differentiably means that for each g in G , the homeomorphism $g(x) = f(g; x)$ of E onto itself is of class C' . It then follows that the functions $f_i(g; x)$ are simultaneously of class C' in $(g; x)$ where (g) is any analytic set of parameters for G .

In the case of $(n-1)$ -dimensional orbits, Theorem A is known so we consider only the $(n-2)$ -dimensional case. Some of the preliminary facts do not depend on differentiability and we proceed at first with only the assumption of a continuous action for G . However we do not know how to carry out the whole proof on this basis and after a certain point we begin to use the assumption of differentiability. The theorem is known for E^3 without differentiability [4; 5] and this special case gives suggestions for the general one. We have also profited from conversations with Leo Zippin, to whom we express thanks.

2. The base space. Throughout this section differentiability is not used. By adding a point at infinity $E = E^n$ becomes the n -sphere $S = S^n$ and the action of G on E may be extended to S by letting every element of G leave p_∞ , the point at infinity, fixed. Thus we may consider the action of G on either E or S according to convenience. (We do not know that G is differentiable at p_∞ and whether coordinates for S can be chosen to make G everywhere differentiable on S is unknown in general. Of course Theorem A will imply for our case that differentiable coordinates can be chosen everywhere, but this is not known in advance.) Associated with E and S there are the orbit spaces E^* and S^* whose "points" are the orbits of G . The map from E to E^* or S to S^* is denoted by T , and T is continuous and open. Both E^* and S^* are connected and locally connected.

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Let X denote the set of points in S whose orbits have dimension $n-2$ and let

$$B = S - X.$$

Then [6] $\dim B \leq n-2$ and X is an open connected set. Every point of B is accessible because of the fact that a closed $(n-2)$ -dimensional set cannot separate an n -dimensional manifold even locally. Thus if $b \in B$ there is an arc bc which lies in X except for the end point b , and a corresponding statement for the base space follows.

Let C be the subset of X where the stability group G_x (=elements of G leaving x fixed) is continuous and let D be the subset of X where G_x is discontinuous.

If x is any point of S there is a neighborhood V of x such that if y is in V then G_y is conjugate to a subgroup of G_x [5]. If x and y are any two points of X then

$$\dim G_x = \dim G_y = \dim G - (n-2).$$

Therefore if x is in X and y in X is near to x then the identity components of G_x and G_y , denoted by G_x^* and G_y^* , are conjugate; because X is connected it follows that G_x^* and G_y^* are conjugate for any two points x and y in X . The set D is closed relative to X .

If a^*b^* is any arc in C^* then there is a cross-sectioning arc ab in the set $T^{-1}(a^*b^*)$. This follows from the fact that such cross-sectioning arcs exist locally [3] and that these local cross-sections can be pieced together.

Also this cross-section may be so chosen that G_x is constant on the cross-section. In order to see this choose some point y in $T^{-1}(a^*b^*)$ and let N be the normalizer of G_y . If E is the set of those points in $T^{-1}(a^*b^*)$ left fixed by G_y , then E is invariant under N ; and also any orbit of $N(z)$ in E is such that

$$G(z) \cap E = N(z).$$

Now cross-sections for N (which will also be cross-sections for G) may be found locally and on these G_x is constant. These small cross-sections may be translated by elements of N to join with others to make a cross-section for $T^{-1}(a^*b^*)$ and on it G_x is constant.

The above can be strengthened as follows:

LEMMA 1. *If J^* is an arc a^*b^* in X^* which lies in C^* except possibly for a^* or b^* or both, then there is a cross-sectioning arc ab in $T^{-1}(a^*b^*)$, and it can be so chosen that G_x is constant on $T^{-1}(a^*b^* - (a^* \cup b^*))$.*

We omit the proof.

For any x in S let $m(x)$ be the number of components of G_x . This function considered relative to X is upper semi-continuous. Since $m(x)$ is constant on an orbit there is an associated function $m(x^*)$ defined as S^* .

If a^*b^* is in X^* and

- (1) $m(x^*)$ is constant in $a^*b^* - (a^* \cup b^*)$,
- (2) $m(a^*) = m(b^*) = km(x^*)$, $k \neq 1$, $x^* \in a^*b^* - (a^* \cup b^*)$

then $T^{-1}(a^*b^*)$ carries an $(n-1)$ -dimensional cycle which is non-trivial mod k , but not with integer coefficients. Such a set cannot exist in S by the Alexander duality theorem and therefore an arc of the kind described above cannot exist in X^* .

Suppose now that a^*b^* is an arc in S^* such that

- (1) $a^* \in S^* - X^*$,
- (2) $a^*b^* - (a^* \cup b^*) \subset C^*$,
- (3) $b^* \in D^*$.

An argument of the kind described above shows that $T^{-1}(a^*b^*)$ carries a cycle mod k for some k , but carries no nontrivial integral cycle. As before this proves that no arc of the kind described exists.

From considerations of this kind we can see as in [4] that D^* is vacuous and that $C = X$, $C^* = X^*$.

We can easily see that for any simple closed curve J^* in $C^* = X^*$, $T^{-1}(J^*)$ is a closed manifold of dimension $n-1$. Therefore $T^{-1}(J^*)$ separates X and S and hence J^* separates X^* and S^* . By Lemma 1, for any arc J^* in X^* , $T^{-1}(J^*)$ is topologically the product of an arc with an orbit $G(x)$, $x \in T^{-1}(J^*)$. Therefore $T^{-1}(J^*)$ cannot separate X and hence J^* cannot separate X^* . This fact is needed to show X^* is a two-cell using [8].

Since $\dim B \leq n-2$, it follows that every point of B is accessible from X , that is if $b \in B$, there is an arc bx which lies in X except for the point b . Projecting into S^* , $T(bx)$ becomes a path joining $T(b)$ and $T(x)$ with no point in $T(B)$ except $T(b)$. In the set $T(bx)$ there is an arc joining $T(b)$ and $T(x)$. Hence every point of B^* is accessible from X^* . Similar arguments show that any point of an arc or simple closed curve in X^* is accessible from the complement of the arc or simple closed curve. Accessibility is used in proving Lemmas 2 and 3 below.

We shall see below that X^* is an open two-cell. Since X is a fibre space and has local cross-sections it then will follow that there is a cross-section in the large for X and by methods already suggested it

may be taken that G_x is constant on this cross-section. Hence topologically X is the product of a two-cell and an orbit $G(x)$, $x \in X$. This will imply further by the Alexander duality theorem that B contains an infinite number of points. The point at infinity is an orbit in B but it is now seen that B must contain at least one other orbit. The Alexander duality theorem implies further that B is connected.

LEMMA 2. X^* is an open two-cell.

If J^* is a simple closed curve in X^* , let

$$S - T^{-1}(J^*) = U_1 \cup U_2$$

where U_1 and U_2 are connected, open, and disjoint. By [8], X^* will be an open two-cell provided one of $U_1^* = T(U_1)$ and $U_2^* = T(U_2)$ has a compact closure in X^* while the other does not. To prove this it is sufficient to prove the corresponding fact about U_1 , U_2 , and X . Either U_1 or U_2 contains p_∞ ; suppose it is U_2 .

We must now show that U_1 contains no point of B . This can be done as in the special case $n=3$ [5, p. 266] and will not be done in detail here. Briefly if U_1 contains a point of B then (using accessibility) there is an arc $a^*b^*c^*$ in X^* where $a^* \in B^*$, $c^* \in B^*$, $b^* \in J^*$, the open arc a^*b^* is in $U_1^* - B^*$, and the open arc b^*c^* is in $U_2^* - B^*$. Then $T^{-1}(a^*b^*c^*)$ separates S and X into two connected domains V_1 and V_2 , and $T^{-1}(J^* - b^*)$ must be in one of them, say in V_1 . There cannot be points of V_2 in both U_1 and U_2 and since $T^{-1}(a^*) \cap \overline{V_2} \neq \emptyset$, $T^{-1}(c^*) \cap \overline{V_2} \neq \emptyset$ this is a contradiction.

LEMMA 3. S^* is a closed two-cell and B^* is the simple closed curve boundary.

The proof is as in [4; 5] and we give only a sketch without complete details.

If z^* is a point of B^* we shall prove that $B^* - z^*$ is connected. If not,

$$B^* = U^* \cup Y^*$$

where U^* and Y^* are nonvacuous compact sets intersecting in either z^* or the null set. Let u^* and y^* be points of U^* and Y^* (both different from z^*) and (using accessibility) let u^*y^* be an arc the open part of which is in X^* . Then $T^{-1}(u^*y^*)$ carries an $(n-1)$ -cycle which bounds in the complement of $T^{-1}(z^*)$. Let x^* be an interior point of the arc u^*y^* . Then $T^{-1}(x^*)$ bounds in the complement of Y^* and in the complement of U^* , but it does not bound in the complement of $B^* = U^* \cup Y^*$. This contradicts Corollary W¹ of [1] and proves

that $B^* - z^*$ is connected.

Next if u^* and y^* are any two points of B^* there is an arc u^*y^* as above and using $T^{-1}(u^*y^*)$, it may be shown that the pair u^* and y^* separates B^* . Hence [7] B^* is a simple closed curve. It can now be checked that S^* satisfies all the conditions required to make it a closed two-cell with boundary B^* [7; 8] and this completes the proof of the lemma.

3. The orbits of B . We now begin to use the differentiability. Let a be a point of B such that $G(a)$ has positive dimension, and choose coordinates at a [2] in which G_a acts orthogonally. There exists a cell P which is perpendicular to $G(a)$, invariant under G_a . It may be taken as spherical with center a . Furthermore P may be assumed so small that for any $y \in P$,

$$G(y) \cap P = G_a(y).$$

If C is a small cell in G perpendicular to G_a then there is a natural map

$$C \times P \rightarrow C(P)$$

and this map is a homeomorphism provided P and C are sufficiently small as we shall assume.

For any $p \in P$,

$$G_p \subset G_a.$$

Note that $m(x)$ is constant on $G(x)$ so $m(x^*)$ may be defined.

LEMMA 4. *In B^* let a^*b^* be a closed arc such that*

- (1) *all orbits of $T^{-1}(a^*b^*)$ have the same dimension,*
- (2) *G_x is continuous on $T^{-1}(a^*b^* - (a^* \cup b^*))$. Then for x^* interior to a^*b^* , $m(a^*)/m(x^*)$ is either 1 or 2.*

The lemma can be seen to be true for zero-dimensional orbits so we assume that each orbit of $T^{-1}(a^*b^*)$ has positive dimension. Let a be a point of $T^{-1}(a^*)$. Around a choose coordinates so that G_a acts orthogonally and choose P as indicated above. For x in the interior of $T^{-1}(a^*b^*)$ and in P , G_x is an open subgroup of G_a and this open subgroup will now be denoted by H ; then H also acts orthogonally in the chosen coordinates around a . Let Q be the set which is left fixed by H so that near a , Q is a linear set which includes a . The set $G(x) \cap P$ is finite, for otherwise $\dim G(x)$ would be greater than $\dim G(a)$ contrary to hypothesis.

The linear set $P \cap Q$ contains a and x so that it is at least one-dimensional; on the other hand it cannot be more than one-dimen-

sional and hence $P \cap Q$ is, locally, an interval in the neighborhood of a .

The identity component of H , denoted by H^* , leaves $P \cap Q$ fixed. The fixed point set of H^* in P is a linear set, which would have dimension at least 2 if it contained points not in $P \cap Q$. This is impossible and hence the fixed point set of H^* is precisely $P \cap Q$.

Let g be an element of G_a , and let x be as above. Since H^* is invariant in G_a we have

$$H^*g(x) = gH^*(x) = g(x)$$

and hence

$$Hg(x) = g(x).$$

Thus $P \cap Q$ is invariant under G_a . When a compact group acts on an interval it must be, effectively, either the identity or a reflection. Therefore G_a/H has either one element or two elements and this completes the proof.

LEMMA 5. *Let a^*b^* be an arc in B^* such that all orbits of a^*b^* are of the same dimension. Then there exists a point c^* in $a^*b^* - a^*$ such that $m(x^*)$ is constant on $a^*c^* - a^*$.*

There is some d^* of $a^*b^* - a^*$ such that if $a \in T^{-1}(a^*)$ and $x \in T^{-1}(a^*d^*)$ then G_x is conjugate to an open subgroup of G_a . Take a cell P as described above and let x_n be a sequence of points in $T^{-1}(a^*d^* - a^*) \cap P$ where $\lim x_n = a$ and for each n , G_{x_n} is constant, that is

$$G_{x_n} = H$$

where H is an open subgroup of G_a . Let Q be the set left fixed by H so that as before $P \cap Q$ is locally an interval which is left invariant by G_a . The lemma now follows.

LEMMA 6. *Let a^*c^* be an arc in B^* with an inner point b^* . Suppose all orbits in $T^{-1}(b^*c^*)$ are of the same dimension r and that every orbit in $T^{-1}(a^*b^* - b^*)$ is of dimension $> r$. Then $m(b^*)$ is $2m(x^*)$ for any x^* immediately to the right of b^* .*

This follows from the results above in Lemmas 4 and 5.

The two lemmas below also follow from those above.

LEMMA 7. *Let a^*c^* be an arc in B^* with an inner point b^* . If all the orbits in $T^{-1}(a^*c^*)$ are of the same dimension and $m(x^*)$ is continuous at every point of $a^*c^* - b^*$ then it is continuous at b^* .*

LEMMA 8. *Let a^*c^* be an arc in B^* with an inner point b^* . If all the*

orbits of $T^{-1}(a^*b^*)$ have the same dimension r , then either all those immediately beyond b^* have dimension r or they all have dimension greater than r .

We now see that if a^*b^* is an interval such that all orbits of $T^{-1}(a^*b^* - (a^* \cup b^*))$ have the same dimension then $m(x^*)$ is constant on the interior of a^*b^* .

4. Cross-sections.

LEMMA 9. Let D^* be a closed two-cell in S^* which intersects B^* in an arc a^*b^* such that G_x is relatively continuous on $T^{-1}(a^*b^*)$. Then there is a two-cell D which is a cross-section for $T^{-1}(D^*)$; D can be chosen so that G_x is constant on $D - T^{-1}(a^*b^*)$ as well as on $D \cap T^{-1}(a^*b^*)$.

Let x^* be a point interior to a^*b^* and let x be a point in $T^{-1}(x^*)$. Choose coordinates around x so that G_x acts orthogonally in these coordinates and choose the cell P as described earlier. For P small enough, as will be assumed, each orbit of $T^{-1}(a^*b^*)$ cuts P in precisely one point and this is left fixed by G_x . For any y in P

$$G(y) \cap P = G_x(y).$$

In $T(P)$ there is an open two-cell σ^* whose boundary intersects a^*b^* in an arc including x^* in its interior. In P there is an open two-cell σ which is a cross-section for $T^{-1}(\sigma^*)$ on which G_x is constant. Along $T^{-1}(a^*b^*)$, σ converges to a single point on each orbit. Hence by adding these limit points we can obtain a closed two-cell cross-section of all the orbits of G in the vicinity of x .

We now see that each point of D^* is in an open subset of D^* for which there is a cross-section. Hence D^* is covered by a finite number of such open sets. Furthermore on these local cross-sections we may assume that G_x is constant for the parts corresponding to a^*b^* and for the parts corresponding to $D^* - a^*b^*$.

We now follow a device first used by Seifert. First divide D^* into a finite set of closed two-cells $\sigma_1^*, \dots, \sigma_k^*$ such that

(1) there is a cross-section for each σ_i^* of the kind described; for this we have only to be sure each σ_i^* is in some one of the above open subsets of D^* ;

(2) we always have $\sigma_1^* \cup \dots \cup \sigma_{i-1}^*$ is a two-cell and

$$(\sigma_1^* \cup \dots \cup \sigma_{i-1}^*) \cap \sigma_i^* = J_i^* \text{ is an arc.}$$

Assume now that there is a cross-section for $\sigma_1^* \cup \dots \cup \sigma_{i-1}^*$, which is, say, K_{i-1} . Let the cross-section for σ_i^* be σ_i with J_i corre-

sponding to J_i^* . For the sake of convenience we may assume that σ_i is the product of the closed unit interval I with itself in which J_i is $0 \times I$ and $\sigma_i \cap T^{-1}(a^*b^*)$ is either null or $I \times 0$. We may also assume that $G_x = G_y$ for any $x \in \sigma_i - T^{-1}(a^*b^*)$ and $y \in K_{i-1} - T^{-1}(a^*b^*)$ because we can always replace σ_i by $g\sigma_i$ for a suitable $g \in G$. Let N be the normalizer of G_x , $x \in \sigma_i - T^{-1}(a^*b^*)$, and let H be the quotient group N/G_x . Then for any $x \in J_i - T^{-1}(a^*b^*)$ there is a unique element h_x of H such that $h_x(x) \in K_{i-1}$. Therefore $x \rightarrow h_x$ is a map α of $J_i - T^{-1}(a^*b^*)$ into H . If $J_i \cap T^{-1}(a^*b^*) \neq \emptyset$, it contains a single point, namely $(0, 0)$. As x tends to $(0, 0)$, $\alpha(x)$ converges to a compact subset of H which we denote by $\alpha(0, 0)$. Since $\alpha(0, 0) \cdot (0, 0)$ is a single point, namely $K_{i-1} \cap G(0, 0)$, it follows that each $\alpha(0, 0) \cdot (s, 0)$, $0 \leq s \leq 1$, is a single point. Hence the set

$$\alpha(0, t) \cdot (s, t) \quad \text{for all } (s, t) \in \sigma_i$$

together with K_{i-1} forms a cross-section for $\sigma_1^* \cup \dots \cup \sigma_i^*$. Thus we can, by induction, get the desired cross-section. This completes the proof.

The result of Lemma 9, as well as the method of proof are used in Lemma 10.

LEMMA 10. *In $B - p_\infty$, G_x cannot be continuous everywhere unless B consists only of fixed points. Further, if in $B - p_\infty$, G_x is continuous except at one orbit then this orbit is a fixed point.*

Assume G_x is continuous in $B - p_\infty$ everywhere except possibly at an orbit $G(b)$ where it may or may not be continuous. We shall prove that $G(b)$ is a point which will prove the lemma.

Let $p_\infty^*b^*$ be an arc the interior of which is in X^* . Then $p_\infty^*b^*$ divides S^* into two closed two-cells S_1^* , S_2^* with $S_1^* \cap S_2^* = p_\infty^*b^*$. Let $S_1^* - (p_\infty^* \cup b^*)$ be divided into a countable set of closed two-cells

$$\sigma_n^*, n = 0, \pm 1, \pm 2, \dots,$$

so that

(1) $\sigma_i^* \cap \sigma_j^*$ is an arc or null according as $|i - j| = 1$ or not.

(2) Each σ_n^* satisfies the hypothesis of Lemma 9 and then there is, by Lemma 9, a cross-section σ_n for σ_n^* such that G_x is constant on $\sigma_n - B$ as well as on $\sigma_n \cap B$.

As in the proof of Lemma 9, we may adjust the cross-sections σ_n such that each $\sigma_n \cap \sigma_{n+1}$ is a cross-section for $\sigma_n^* \cap \sigma_{n+1}^*$. To piece these cross-sections σ_n together we get a cross-section D_1 for $S_1^* - (p_\infty^* \cup b^*)$. Similarly there is a cross-section D_2 for $S_2^* - (p_\infty^* \cup b^*)$. Again we may adjust D_2 so that $D_1 \cap D_2$ is a cross-section for $p_\infty^*b^* - (p_\infty^* \cup b^*)$. Hence

we have a cross-section $D = D_1 \cup D_2 \cup p_\infty$ of all orbits in S except for $G(b)$. The set D is, of course, homeomorphic to a closed two-cell with one boundary point removed. If $G(b)$ is not a point a carries an r -cycle mod 2, $0 < r \leq n-2$, and it must be linked with a cycle z , of dimension $n-r-1$, which has a compact carrier Z in $S-G(b)$.

The cross-section D and the known structure of the orbits can now be used to deform Z into any preassigned neighborhood of p_∞ , while remaining in $S-G(b)$. This contradicts the fact that z links $G(b)$ and proves the lemma.

5. Discontinuities in B .

LEMMA 11. *Let a^*b^* be an interval in B^* such that G_x is continuous in $T^{-1}(a^*b^* - (a^* \cup b^*))$. Let a and b be points in $T^{-1}(a^*)$, $T^{-1}(b^*)$, and suppose G_x is discontinuous (relative to $T^{-1}(a^*b^*)$) at $T^{-1}(a^*)$ and $T^{-1}(b^*)$. Then not both a^* and b^* can be different from p_∞^* .*

Assume both a^* and b^* are different from p_∞^* . At $T^{-1}(a^*)$ the dimension of orbits is either lowered or else G_x has a discontinuity of "index" two. A similar remark applies to $T^{-1}(b^*)$. Therefore $T^{-1}(a^*b^*)$ carries a cycle z not bounding in B . This is linked with a cycle y carried by a compact set $Y \subset S-B$. Then Y can be deformed into any preassigned neighborhood of p_∞ . This gives a contradiction and proves the lemma.

We now assume that B contains orbits of dimension >0 (if all orbits of B have dimension zero the proof in the next section is valid). Let r be the highest dimension of any orbit in B and choose $y \in B$ such that $\dim G(y) = r$. Then by Lemmas 5, 7 and 11, $T(y)$ is in an interval a^*b^* such that

- (1) $a^* = T(p_\infty)$,
- (2) G_x is continuous in $T^{-1}(a^*b^* - (a^* \cup b^*))$,
- (3) $T^{-1}(b^*)$ has lower dimension than r .

In the remainder of B let s be the highest dimension of any orbit so that $s \leq r$. Let z be a point in $B - T^{-1}(a^*b^*)$ such that

$$\dim G(z) = s.$$

We see first that $T(z)$ is in an interval a^*c^* with G_x continuous on the interior of $T^{-1}(a^*c^*)$. If $c^* \neq b^*$ there is an interval $c_1^*b_1^*$ between c^* and b^* (the part not including a^*) with G_x constant on the interior and with orbits at the ends having lower dimension than on the interior. This has been shown to be impossible. Hence $c^* = b^*$. Then $T(z)$ is in a second interval a^*b^* (the union of the two intervals is B^*) where G_x is continuous on the interior. We see by Lemma 10 that b^* corresponds to a fixed point also. Thus a^* and b^* correspond to fixed

points and divide B^* into two open intervals; on the inverse of each open interval G_x is continuous.

6. Proof of Theorem A. As seen in the proof of Lemma 10, we can now construct a closed two-cell C in S^n which is a cross-section in the large. Using $C - p_\infty$, which is topologically an open half-plane which cross-sections E , it can be seen that there is a natural homeomorphism of an open invariant neighborhood of the fixed point in $E = E^n$ onto all of E^n , and that this natural homeomorphism preserves orbits. By the theorem of Bochner, the group G can be assumed to act linearly in a neighborhood of the fixed point. This and the existence of the homeomorphism described above show that G acts linearly in all of E^n if coordinates are properly chosen. This completes the proof of Theorem A.

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INSTITUTE FOR ADVANCED STUDY,
UNIVERSITY OF MICHIGAN, AND
UNIVERSITY OF ILLINOIS