SIMPLE NEAR-RINGS OF DIFFERENTIABLE TRANSFORMATIONS

D. W. BLACKETT

An earlier paper [1] reduced the study of semisimple near-rings to the study of simple near-rings. The definition of a radical for near-rings by W. E. Deskins [2] permits a partial description of many non-semisimple near-rings in terms of simple near-rings. The present paper explores the structure of some simple near-rings.

Represent vectors of an n -dimensional vector space V over the real numbers by their coordinates

$$X = (x_1, \dots, x_n), \quad Y = (y_1, \dots, y_n), \quad A = (a_1, \dots, a_n), \dots$$

with respect to a fixed basis of V . A transformation $T: X \rightarrow XT = Y$ of V into itself may be described by n coordinate functions

$$y_i = f^i(x_1, \dots, x_n) = f^i(X)$$

or by the n -tuple

$$Y = XT = F(X) = (f^1(X), \dots, f^n(X)).$$

The i th partial derivative of $F(X)$ at A is

$$F_i(A) = (f_i^1(A), \dots, f_i^n(A))$$

where $f_i^j(A)$ is the i th partial derivative of $f^j(X)$ evaluated at A . The transformation T is differentiable at A if the n partial derivatives of $F(X)$ exist in a neighborhood of A and are continuous at A . The total differential at A of a transformation T which is differentiable at A is

$$\Delta F(A) = F_1(A)\Delta x_1 + \dots + F_n(A)\Delta x_n.$$

THEOREM 1. *Let N be a near-ring (under addition and iteration) of transformations of V which have origin θ of V as a fixed point and also as a point of differentiability. The correspondence*

$$T \rightarrow \begin{pmatrix} f_1^1(\theta) & \dots & f_1^n(\theta) \\ \vdots & & \vdots \\ f_n^1(\theta) & \dots & f_n^n(\theta) \end{pmatrix} = M(T)$$

is a near-ring homomorphism of N into a ring of $n \times n$ matrices over the

Presented to the Society, February 25, 1956; received by the editors July 12, 1955.

real numbers. The kernel of this homomorphism is the two-sided ideal of N consisting of all transformations in N with vanishing total differential at θ .

Let T and S be transformations in N described by $F(X)$ and $G(X)$. The linearity of differentiation insures

$$M(T + S) = M(T) + M(S).$$

That

$$M(TS) = M(T)M(S)$$

follows because the j th partial derivative of $g^i(F(X))$ at θ is

$$\sum_{k=1}^n g_k^i(\theta) f_{j,i}^k(\theta).$$

Thus the correspondence $T \rightarrow M(T)$ is a near-ring homomorphism. The kernel of this homomorphism consists of those transformations in N with vanishing total differential at the origin. Therefore these transformations form a two-sided ideal of N .

COROLLARY 1. *The ring of linear transformations of V is a maximal simple near-ring in the near-ring of all transformations of V which have θ as a fixed point and as a point of differentiability.*

Let N be a near-ring of transformations which have θ as a fixed point and as a point of differentiability. Suppose N contains all the linear transformations and at least one nonlinear transformation T . If L is the linear transformation with matrix $M(T)$, $T - L$ is a nonlinear and, hence, nonzero transformation in N with vanishing total differential at θ . Theorem 1 now implies that N has a proper two-sided ideal and is not simple.

THEOREM 2. *Let N be a simple near-ring of transformations of V which have θ as a fixed point, are differentiable at θ , and include some transformation with nonvanishing total differential at θ . If X is a vector of V , the closure $\text{Cl}(XN)$ of the set of vectors XT for T in N is a subspace of V invariant under N . If $\text{Cl}(XN) \neq \theta$, the transformations in N restricted to $\text{Cl}(XN)$ faithfully represent N as a simple subring of the ring of linear transformations of $\text{Cl}(XN)$.*

N can be simple only if the homomorphism $T \rightarrow M(T)$ described in Theorem 1 is an isomorphism. Therefore N is a ring. Because N is a simple ring of characteristic zero, there is, for each transformation T in N and any integers p and q ($\neq 0$), a transformation $(p/q)T$ in

N such that

$$X\left(\frac{p}{q}T\right) = \frac{p}{q}(XT)$$

for all X in V . Hence if Y is in $\text{Cl}(XN)$ and μ is a real number, μY is also in $\text{Cl}(XN)$. That $\text{Cl}(XN)$ is closed under addition follows because $X(T+S) = XT + XS$. This shows that $\text{Cl}(XN)$ is a subspace of V .

Since N is a ring

$$(1) \quad (T + S)R = TR + SR$$

for all T, S, R in N . In particular

$$(pT)R = (T + \cdots + T)R = TR + \cdots + TR = p(TR).$$

Hence

$$p(XTR) = (p(XT))R = \left(\frac{pq}{q}(XT)\right)R = q\left[\left(\frac{p}{q}(XT)\right)R\right]$$

or

$$(2) \quad \left(\frac{p}{q}(XT)\right)R = \frac{p}{q}(XTR)$$

for all X in V and T, R in N .

Suppose

$$Y = \lim_{m \rightarrow \infty} XT_m.$$

For a transformation R in N select a nonzero rational number p/q such that R is continuous at $(p/q)Y$. This is possible because R is differentiable and, hence, continuous in a neighborhood of θ . Now

$$\begin{aligned} \frac{p}{q} \lim_{m \rightarrow \infty} (XT_m R) &= \lim_{m \rightarrow \infty} \left(\frac{p}{q} XT_m R\right) = \left(\lim_{m \rightarrow \infty} \frac{p}{q} XT_m\right)R \\ &= \left(\frac{p}{q} \lim_{m \rightarrow \infty} XT_m\right)R = \frac{p}{q} YR \end{aligned}$$

and

$$\lim_{m \rightarrow \infty} XT_m R = YR.$$

Hence R is continuous on $\text{Cl}(XN)$. Since $X(T_m R)$ is in $\text{Cl}(XN)$, YR

is in $\text{Cl}(XN)$. It follows that $\text{Cl}(XN)$ is an invariant subspace of V with respect to N .

The continuity of R on $\text{Cl}(XN)$ together with (2) shows that

$$(3) \quad (aY)R = a(YR)$$

for a real, Y in $\text{Cl}(XN)$, and R in N . By (1)

$$(XT + XS)R = XTR + XTS$$

for X in V and T, S, R in N . From the continuity of R on $\text{Cl}(XN)$

$$(4) \quad (Y + Z)R = YR + ZR$$

for any Y, Z in $\text{Cl}(XN)$. By (3) and (4) the transformations in N restricted to $\text{Cl}(XN)$ are linear. Since the restriction of the transformations in N to $\text{Cl}(XN)$ effects a homomorphism of N , the simplicity of N implies that this restriction gives a faithful representation of N if $\text{Cl}(XN) \neq \emptyset$.

The following example shows that the conclusion of linearity over $\text{Cl}(XN)$ cannot be strengthened to linearity over V .

EXAMPLE 1. Let V be two-dimensional and define the transformation T_a for each real number a by

$$y_1 = x_1 f(x_2) a, \quad y_2 = 0$$

where $f(x)$ is a fixed real function with $f(0) = 1$ and $f'(x)$ continuous at 0. These transformations form a simple near-ring isomorphic to the field of real numbers. No T_a ($a \neq 0$) is linear over V unless $f(x)$ is constant.

EXAMPLE 2. Let V be one-dimensional and define for each real number a the transformation

$$S(a): x \rightarrow y = |x| a.$$

These transformations form a simple near-ring with the multiplication law

$$S(a)S(b) = S(|a| b).$$

This example demonstrates the necessity of the differentiability assumption of Theorem 2.

EXAMPLE 3. Let V and $S(a)$ be as in Example 2 and introduce the linear transformations

$$T(a): x \rightarrow y = xa.$$

The transformations of the form $T(a) + S(b)$ form a near-ring N with the multiplication rules

$$[T(a) + S(b)]T(c) = T(ac) + S(bc)$$

and

$$\begin{aligned} [T(a) + S(b)]S(c) &= S(|a|c) + T(|b|c) \quad \text{if } ab \geq 0 \text{ and } |a| \geq |b|, \\ &= T(|a|c) + S(|b|c) \quad \text{if } ab \geq 0 \text{ and } |a| \leq |b|, \\ &= S(|a|c) - T(|b|c) \quad \text{if } ab \leq 0 \text{ and } |a| \geq |b|, \\ &= -T(|a|c) + S(|b|c) \quad \text{if } ab \leq 0 \text{ and } |a| \leq |b|. \end{aligned}$$

These rules are derived from the simplification of

$$|xa + |x|b|$$

in the various cases.

LEMMA 1. *A nontrivial right module of N consists of all elements $T(ar) + S(a)$ where r is a fixed real number with $-1 \leq r \leq 1$ and a runs through all real numbers.*

If

$$T(ar_1) + S(a) \quad \text{and} \quad T(br_2) + S(b) \quad (ab \neq 0)$$

are elements of a right module R of N , R contains

$$[T(ar_1) + S(a)]T(b) - [T(br_2) + S(b)]T(a) = T(abr_1 - bar_2).$$

If $r_1 \neq r_2$ and $\Delta = abr_1 - bar_2$,

$$T(\Delta)T(c/\Delta) = T(c)$$

is in R for every c . Since $T(ar_1) + S(a)$ and $T(ar_1)$ are in R , $S(a)$ is in R . Hence

$$S(a)T(d/a) = S(d)$$

is in R for every d . Therefore $T(c) + S(d)$ is in R for all c and d , and $R = N$.

If $T(ar) + S(a)$ is in R and $|r| > 1$, R contains

$$[T(ar) + S(a)]T(|ar|) = T(ar|ar|) + S(a|ar|)$$

and

$$\begin{aligned} [T(ar) + S(a)]S(a) &= S(|ar|a) \pm T(|a|a) \\ &\quad (+ \text{ if } r \geq 0 \quad \text{and} \quad - \text{ if } r \leq 0). \end{aligned}$$

By subtraction

$$T(ar|ar|) \pm |a|a \neq T(0)$$

is in R . As above $R = N$.

If $T(ar) + S(a)$ ($a \neq 0$) is in R , R contains

$$[T(ar) + S(a)]T(b/a) = T(br) + S(b)$$

for all b . This combined with the previous facts shows that the only sets which could be nontrivial right modules of N are those described in the lemma. Direct computation shows that these sets are right modules.

LEMMA 2. N is a simple near-ring.

From Lemma 1 the intersection of two distinct nontrivial right modules of N is $T(0)$. Therefore the right modules satisfy the descending chain condition. If R is the right module of elements

$$\begin{aligned} & T(ar) + S(a) \quad r \text{ fixed with } |r| \leq 1, \\ & [T(ar) + S(a)][T(br) + S(b)] \\ &= T(abr^2) + S(abr) \pm T(|ar|b) + S(|a|b) \quad (+ \text{ if } r \geq 0, - \text{ if } r \leq 0) \\ &= T(abr^2 \pm |ar|b) + S(abr + |a|b). \end{aligned}$$

If $b \neq 0$, $abr - |a|b$ cannot be zero for both positive and negative values of a . Hence the product of nonzero elements of R is not always zero, and R is not nilpotent. If $b < 0$,

$$S(b)[T(ar) + S(a)] = S(bar) + S(|b|a)$$

is not in R . Therefore R is not a left ideal. Thus there are no proper two-sided ideals, and N is simple.

This example shows the necessity of the differentiability assumption in Theorem 2 and in Corollary 1.

EXAMPLE 4. Suppose G is an additive subgroup of V and V^0 is a subset of V with $V^0 \cap G = \theta$. Define the transformation T_g for each g in G by

$$\begin{aligned} XT_g &= g & \text{if } X \text{ is not in } V^0, \\ XT_g &= \theta & \text{if } X \text{ is in } V^0. \end{aligned}$$

The transformations T_g form a simple near-ring with the multiplication rules

$$T_g T_{g'} = T_{g'} \quad (g \neq \theta)$$

and

$$T_\theta T_{g'} = T_\theta.$$

If G is a discrete group V^0 may contain a neighborhood of θ , so that

each T_θ is differentiable at θ and has total differential zero at θ . This shows the necessity of the hypothesis of Theorem 2 that there be at least one function in N with nonvanishing total differential at θ .

The m th total differential at A of a transformation T described by $F(X)$ is the n -tuple

$$\Delta^m F(A) = \left(\sum_1^n \cdots \sum_1^n f_{i_1 \cdots i_m}^1(A) \Delta x_{i_1} \cdots \Delta x_{i_m}, \right. \\ \left. \cdots, \sum_1^n \cdots \sum_1^n f_{i_1 \cdots i_m}^n(A) \Delta x_{i_1} \cdots \Delta x_{i_m} \right)$$

where $f_{j_1 \cdots j_m}^i(A)$ is the m th partial derivative of $f^i(X)$ with respect to $x_{j_1}, \cdots, x_{j_m}$ evaluated at A . If all these partial derivatives exist in a neighborhood of A and are continuous at A , then T is m -times differentiable at A .

THEOREM 3. *Let N be a near-ring of transformations of V which have θ as a fixed point, are m -times differentiable at θ ($m > 1$), have vanishing first $m-1$ total differentials at θ , and include some transformation with nonvanishing m th total differential at θ . Under these hypotheses N is not simple.*

The linearity of differentiation establishes that the transformations with vanishing m th total differential at θ are an additive subgroup of N . That they form a two-sided ideal of N follows because any product of two transformations in N has vanishing m th total differential at θ . If this ideal is not the zero ideal, N is not simple. If only the zero transformation has vanishing m th total differential at θ , every product of transformations in N is zero so that N is nilpotent and not simple.

Example 4 shows that the existence of a transformation with nonvanishing m th total differential at θ is a necessary hypothesis in Theorem 3.

A transformation T represented by $F(X)$ is analytic over V if each coordinate function $f^i(X)$ can be expanded in a power series in $x_1, \cdots, x_n$ which converges throughout V .

COROLLARY 2. *Every simple near-ring of analytic transformations of V with θ as a fixed point is a simple ring which is faithfully represented as a ring of linear transformations by restricting the domain of the analytic transformations to the invariant subspace of V generated by a single nonzero vector not annihilated by N .*

The analyticity of the transformations and Theorem 3 show that

the simple near-ring contains a function with nonvanishing (first) total differential at θ . The corollary follows by Theorem 2.

COROLLARY 3. *The simple (nonzero) near-rings of entire functions over the complex plane which vanish at 0 are the rings of transformations described by equations*

$$w = az$$

where a ranges over a subfield of the complex numbers.

If the complex plane is regarded as a two-dimensional space V over the real numbers, the entire functions are analytic transformations in the sense just defined. By Corollary 2 the transformations of the simple near-ring are linear on a (nonzero) subspace of V . From the complex analyticity of the functions, they are linear throughout V . Corollary 3 follows.

REFERENCES

1. D. W. Blackett, *Simple and semisimple near-rings*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 772-785.
2. W. E. Deskins, *A radical for near-rings*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 825-827.

BOSTON UNIVERSITY