COLLECTIVE SINGULARITIES OF A FAMILY OF ANALYTIC FUNCTIONS¹

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Let $\mathfrak{F}$ denote a family of functions and let $\mathcal{R}$ denote the radius of the greatest circle with center at z_0 in which each function is holomorphic and the family is normal. We shall call $\mathcal{R}$ the radius of regularity of $\mathfrak{F}$ at z_0 . Mandelbrojt has stated recently the following formula for computing $\mathcal{R}$ in the case where $\mathfrak{F}$ is subuniformly bounded in $|z - z_0| < \mathcal{R}$ (i.e. uniformly bounded on each compact subset of $|z - z_0| < \mathcal{R}$).

$$\limsup_{n \rightarrow \infty} \sup_{f \in \mathfrak{F}} |a_n(f)|^{1/n} = \frac{1}{\mathcal{R}},$$

where $f(z) = \sum_{n=0}^{\infty} a_n(f)(z - z_0)^n$.

It may be observed that the formula is in form the same as the Hadamard formula for the radius of convergence of a Taylor series. From the definition of $\mathcal{R}$, however, one must consider here the presence of irregular points of a family of functions as well as singular points of a function. In two notes [4] Mandelbrojt presented a composition theorem in which irregular points play a part analogous to that of the singular points in the composition theorem of Hadamard [3]. Montel [6] calls an irregular point of a family a collective singular point. The theorem presented in this paper demonstrates further the analogy between singular point and collective singular point.

D will be called a domain of regularity of $\mathfrak{F}$ if each function is holomorphic and the family is normal in D . A point z' on the boundary of D will be called a regular point of $\mathfrak{F}$ if there is a neighborhood $|z - z'| < \rho$ such that the union of the neighborhood and D is a domain of regularity. Otherwise z' will be called a singular point of $\mathfrak{F}$. The theorem is a generalization of the gap theorem of Hadamard [2]. The method of proof is due to Faber [1] and Mordell [5].

THEOREM 1. *Let $\mathfrak{F}$ be a family of functions $f(z) = \sum_{n=0}^{\infty} a_n(f)z^n$ which has radius of regularity $\mathcal{R}(\mathfrak{F}) = 1$ and is subuniformly bounded in $|z| < 1$. Suppose that to each function there corresponds a sequence of integers $\{\lambda_p(f)\}_{p=1}^{\infty}$ such that $a_n(f) = 0$ if $n \neq \lambda_p(f)$, $p = 1, 2, \dots$. More-*

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over suppose $\lambda_{p+1}(f)/\lambda_p(f) \geq \lambda > 1$, where λ is independent of f . Then each point of $|z| = 1$ is a singular point of $\mathcal{F}$.

PROOF. Choose an integer q such that $q > 1/(\lambda - 1)$ and let $z = (w^q + w^{q+1})/2$. Consider the family $\mathcal{G}$ of functions

$$(1) \quad g(w) = f\left(\frac{w^q + w^{q+1}}{2}\right) = \sum_{n=0}^{\infty} a_n(f) \left(\frac{w^q + w^{q+1}}{2}\right)^n = \sum_{m=0}^{\infty} b_m(g) w^m.$$

Now $|w| \leq 1$, $w \neq 1$ implies $|z| < 1$ so that $\mathcal{R}(\mathcal{G}) \geq 1$ and each point of $|w| = 1$ is a regular point of $\mathcal{G}$ except perhaps $w = 1$. It will be shown that $\mathcal{R}(\mathcal{G}) = 1$ implying $w = 1$ is a singular point of $\mathcal{G}$ and hence $z = 1$ is a singular point of $\mathcal{F}$. The family of functions $\sum_{n=0}^{\infty} a_n(f) e^{in\theta} z^n$ also satisfies the hypothesis so that $z = e^{i\theta}$ is a singular point of $\mathcal{F}$.

It will be sufficient to show that $\mathcal{R}(\mathcal{G}) \leq 1$. The inequality of the hypothesis together with the choice of q result in the fact that no two polynomials

$$\left(\frac{w^q + w^{q+1}}{2}\right) \lambda_l(f) \text{ and } \left(\frac{w^q + w^{q+1}}{2}\right) \lambda_k(f), \quad l \neq k,$$

have common powers of w . Thus each coefficient $b_m(g)$ may be expressed in terms of a single coefficient $a_n(f)$; specifically

$$b_{qn+j}(g) = \frac{a_n(f)}{2^n} C_{j,n}, \quad 0 \leq j \leq n.$$

In particular setting $m_n = qn + [n/2]$

$$b_{m_n}(g) = \frac{a_n(f)}{2^n} C_{[n/2], n}$$

where $[n/2]$ denotes the integral part of $n/2$. Then using the fact that

$$\lim_{n \rightarrow \infty} \left(\frac{C_{[n/2], n}}{2^n} \right)^{1/m_n} = 1$$

and from $\mathcal{R}(\mathcal{F}) = 1$ that $\limsup_f \sup_f |a_n(f)|^{1/n} = 1$, we have

$$\limsup_{n \rightarrow \infty} \sup_{g \in \mathcal{G}} |b_{m_n}|^{1/m_n} = \limsup_{n \rightarrow \infty} \sup_{f \in \mathcal{F}} \left| a_n(f) \frac{C_{[n/2], n}}{2^n} \right|^{1/m_n} = 1.$$

Hence

$$\frac{1}{\mathcal{R}(\mathcal{G})} = \limsup_{m \rightarrow \infty} \sup_{g \in \mathcal{G}} |b_m(g)|^{1/m} \geq 1$$

which was to be proved.

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