CHARACTERIZATIONS OF ALMOST PERIODIC TRANSFORMATION GROUPS¹

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Let X be a uniform space, let T be a multiplicative topological group, and let T act as a transformation group on X .

A subset A of T is said to be (*left*) *syndetic* provided that $T = AK$ for some compact subset K of T . The transformation group T is said to be *almost periodic on X* provided that if α is an index of X , then there exists a syndetic subset A of T such that $x\alpha \subset xA$ for all $x \in X$. If $x \in X$, then the transformation group T is said to be *locally almost periodic at x* provided that if U is a neighborhood of x , then there exist a neighborhood V of x and a syndetic subset A of T such that $VA \subset U$. The transformation group T is said to be *locally almost periodic on X* in case T is locally almost periodic at every point of X . If $x \in X$, then the transformation group T is said to be *locally weakly almost periodic at x* provided that if U is a neighborhood of x , then there exist a neighborhood V of x , a syndetic subset A of T , and a compact subset C of T such that $y \in V$ implies the existence of a subset B of T for which $A \subset BC$ and $yB \subset U$. It is readily proved that if $x \in X$, then T is locally weakly almost periodic at x if and only if for each neighborhood U of x there exist a neighborhood V of x and a compact subset K of T such that $VT \subset UK$.

If $x \in X$, then the transformation group T is said to be *equicontinuous at x* provided that if α is an index of X , then there exists an index β of X such that $x\beta t \subset x\alpha$ for all $t \in T$. The transformation group T is said to be *equicontinuous on X* in case T is equicontinuous at every point of X . The transformation group T is said to be *uniformly equicontinuous on X* provided that if α is an index of X , then there exists an index β of X such that $x\beta t \subset x\alpha$ for all $x \in X$ and all $t \in T$. It is readily verified that if X is compact, then T is uniformly equicontinuous if and only if T is equicontinuous.

The transformation group T is said to be *distal on X* provided that if $x, y \in X$ with $x \neq y$, then there exists an index α of X such that $(xt, yt) \notin \alpha$ for all $t \in T$.

We also consider T to be a transformation group acting on $X \times X$ in the following manner: if $x, y \in X$ and if $t \in T$, then $(x, y)t$ is defined to be (xt, yt) .

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As a general reference for the notions occurring in this paper, see [2].

LEMMA 1. *Let X be compact and suppose there exists $x \in X$ such that T is locally weakly almost periodic at (x, x) but T is not equicontinuous at x . Then T is not distal on X .*

PROOF. Let $\mathfrak{N}$ be the neighborhood filter of x . Since T is not equicontinuous at x , there exists an open neighborhood U of the diagonal of $X \times X$ such that $(N \times N)T \not\subset U$ for every $N \in \mathfrak{N}$. Define $\mathfrak{F} = \{(N \times N)T \cap U' \mid N \in \mathfrak{N}\}$ where U' is the complement of U in $X \times X$. In order to show that T is not distal on X it is enough to show that $\bigcap \mathfrak{F} \neq \emptyset$. Since $\bar{\mathfrak{F}} = \{\bar{F} \mid F \in \mathfrak{F}\}$ is a closed filter-base on the compact set U' , it follows that $\bigcap \bar{\mathfrak{F}} \neq \emptyset$. We complete the proof by showing that each member of $\mathfrak{F}$ contains some member of $\bar{\mathfrak{F}}$ whence $\bigcap \mathfrak{F} \supset \bigcap \bar{\mathfrak{F}}$ and thus $\bigcap \mathfrak{F} \neq \emptyset$. Let $N \in \mathfrak{N}$. Choose a closed neighborhood N_1 of x such that $N_1 \subset N$. Since T is locally weakly almost periodic at (x, x) , there exist a neighborhood M of x and a compact subset K of T such that $(M \times M)T \subset (N_1 \times N_1)K$ whence $\text{cls } (M \times M)T \subset (N_1 \times N_1)K \subset (N \times N)T$ and $(N \times N)T \cap U' \supset \text{cls } ((M \times M)T \cap U')$. This shows that every member of $\mathfrak{F}$ contains some member of $\bar{\mathfrak{F}}$. The proof is completed.

THEOREM 1. *Let X be compact. Then the following statements are pairwise equivalent:*

- (1) *T is almost periodic.*
- (2) *T is locally almost periodic on X and T is distal on X .*
- (3) *T is locally weakly almost periodic at every point of the diagonal of $X \times X$ and T is distal on X .*

PROOF. It is known [2, 4.38] that if X is compact, then T is almost periodic if and only if T is uniformly equicontinuous. It follows easily from this theorem that (1) implies (2). An independent proof that (1) implies (2) may also be given.

Suppose now that T is locally almost periodic on X and let $x \in X$. We show T is locally weakly almost periodic at (x, x) . Let U be a neighborhood of x . There exist a neighborhood V of x and a syndetic subset A of T such that $VA \subset U$. Let K be a compact subset of T for which $T = AK$. We conclude that $(V \times V)T \subset (U \times U)K$.

It is now clear that (2) implies (3). That (3) implies (1) is immediate from Lemma 1. The proof of the theorem is completed.

We say that X is a *minimal orbit-closure under T* or simply that X is *minimal under T* in case $\bar{xT} = X$ for all $x \in X$. A *discrete flow* is a transformation group whose phase group T is the additive group of

integers with the discrete topology. A discrete flow is completely determined by a homeomorphism of the phase space X onto itself.

None of the conditions in (2) or (3) of Theorem 1 is redundant. It is known [2, 12.63] that there exist compact metrizable zero-dimensional locally almost periodic minimal orbit-closures under discrete flows which are not almost periodic. A simple example of a ring of concentric circles rotating at different rates about their common center shows that distal alone does not imply almost periodicity.

Let $x, y \in X$. The pair (x, y) is said to be *proximal under T* provided that if α is an index of X , then there exists $t \in T$ such that $(xt, yt) \in \alpha$. Of course, there exists a pair of distinct points of X which is proximal under T if and only if T is not distal on X . The pair (x, y) is said to be *syndetically proximal under T* provided that if α is an index of X , then there exists a syndetic subset A of T such that $(xa, ya) \in \alpha$ for all $a \in A$.

LEMMA 2. *Let X be compact, let $x, y \in X$, let (x, y) be proximal under T , let α be an index of X , and let K be a compact subset of T . Then there exists $t \in T$ such that $(xtk, ytk) \in \alpha$ for all $k \in K$.*

PROOF. Since $X \times K$ is compact, the phase projection $\pi: X \times T \rightarrow X$ defined by $\pi(x, t) = xt$ is uniformly continuous on $X \times K$. Hence there exists an index β of X such that $(x_1, x_2) \in \beta$ implies $(x_1k, x_2k) \in \alpha$ for all $k \in K$. Choose $t \in T$ such that $(xt, yt) \in \beta$. The conclusion follows.

LEMMA 3. *Let X be compact, let $x, y \in X$, and let T be locally almost periodic at x . Then (x, y) is syndetically proximal if and only if (x, y) is proximal.*

PROOF. The necessity is obvious. We prove the sufficiency. Suppose (x, y) is proximal. Let α be an index of X . Choose a neighborhood U of x such that $U \times U \subset \alpha$. There exist a neighborhood V of x and a syndetic subset A of T such that $VA \subset U$. Choose an index β of X such that $x\beta^2 \subset V$. There exists a syndetic subset B of T such that $xB \subset x\beta$. Let H be a compact subset of T such that $T = BH$. By Lemma 2 there exists $t_0 \in T$ such that $(xt_0h^{-1}, yt_0h^{-1}) \in \beta$ for all $h \in H$. Now $t_0 = b_0h_0$ for some $b_0 \in B$ and some $h_0 \in H$. Since $t_0h_0^{-1} = b_0$, it follows that $(xb_0, yb_0) \in \beta$. Altogether we now have $xb_0 \in x\beta \subset V$ and $yb_0 \in xb_0\beta \subset x\beta^2 \subset V$ whence $xb_0, yb_0 \in V$ and $(xb_0a, yb_0a) \in U \times U \subset \alpha$ for all $a \in A$. Thus $t \in b_0A$ implies $(xt, yt) \in \alpha$. Since b_0A is a syndetic subset of T , the proof is completed.

THEOREM 2. *Let X be compact and let T be locally almost periodic on X . Then there exists a pair of distinct points of X which is syndetically proximal under T if and only if T is not equicontinuous on X .*

PROOF. Use Theorem 1 and Lemma 3.

Let $x, y \in X$. The pair (x, y) is said to be (*totally*) *asymptotic under* T provided that if α is an index of X , then there exists a compact subset K of T such that $(xt, yt) \in \alpha$ for all $t \in T - K$.

A study (see below) of the example in [1] will reveal that the phrase "syndetically proximal" in Theorem 2 cannot be replaced by "asymptotic," even though it is assumed in addition that X is a compact plane set, T is a discrete flow, X is minimal under T , and T is regularly almost periodic at some points of X .

We indicate briefly why this example cannot possess even a unilaterally asymptotic pair of distinct points. We adopt here the notation used in [1] and we assume familiarity with the paper. First of all, T is locally almost periodic since X is minimal under T and some points of X are regularly almost periodic. This is an application of a general theorem (cf. [2; 5.24]). Since f has equicontinuous powers, no two components of X are even proximal. Consequently, any asymptotic pair of points belonging to X would have to both lie in the same component. Now the nondegenerate components cannot have lengths which tend to zero under iteration of T . This is shown as follows:

Let x be the point $(3^0, 3^0 + 3^1, 3^0 + 3^1 + 3^2, \dots)$ of A . The component V of X which lies over x is the longest and indeed has length 1. Let $\alpha_0, \dots, \alpha_n$ be an arbitrary finite sequence made up of 0, 1, 2. If the map f is applied $(\alpha_0 - 1)3^0 + (\alpha_1 - 1)3^1 + \dots + (\alpha_n - 1)3^n$ times to x , then the point $(\alpha_0 3^0, \alpha_0 3^0 + \alpha_1 3^1, \dots)$ is obtained. Consequently the images of V constitute all nondegenerate components of X and are of length $1/2$ infinitely often in both directions.

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