ON CANONICAL CONSTRUCTIONS. III

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In the first paper in this series a structure S was called canonically reconstructible from a structure T provided S was canonically isomorphic to a structure T' derived from T . The principle proposition of the paper was the scholium which asserted: If a structure S contains a substructure T and if S is canonically reconstructible from T , then every automorphism of T may be extended uniquely to an inner automorphism of S (relative to the derivation of S from T). To illustrate the concepts it was proved in that paper that every automorphism of the group of permutations $\Sigma(N)$ of a set N (finite or infinite) was inner in the group theoretic sense except when N has 6 elements. Let $A(N)$ now denote the subgroup of $\Sigma(N)$ generated by all 3-cycles, i.e., all permutations moving exactly 3 elements of N . This will be called the alternating group on N . It will here be shown that the set N can be canonically reconstructed from any subgroup G of $\Sigma(N)$ which contains $A(N)$ when N has 4, 5, or more than 6 elements. It follows by the principles of *On canonical constructions*, I, that every automorphism of G can then be extended uniquely to an automorphism, i.e., permutation, of N , which in turn induces an inner automorphism of $\Sigma(N)$. This is trivially true when N has 1 or 2 elements. Therefore, it is the case, except when N has cardinality 3 or 6, that every automorphism of G can be extended uniquely to an inner automorphism of $\Sigma(N)$, which is the principle theorem of this paper.¹

Substructures as derived structures. A substructure R of a structure S is construable as a structure derived from S if it is canonically characterisable within S ; such a substructure will be called *characteristic*. The characterization may be given by a statement of the form, " R is the unique substructure of S with the property. . . ." For example, when N is a finite set $A(N)$ may be considered as a structure derived from $\Sigma(N)$, being the unique subgroup of index two. (This gives an example of a derived structure which is a group but in which an inner automorphism as a derived structure is not necessarily inner in the group theoretic sense.) The object of this sec-

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¹ When N has 3 elements the extension is possible but not unique. For finite N the theorem is due to O. Hölder [3].

tion is to demonstrate that for any finite or infinite N , if G is a subgroup of $\Sigma(N)$ which contains $A(N)$ then $A(N)$ may be considered as a structure derived from G . Let n henceforth denote the cardinality of N . The matter is already settled for finite n . For $n \geq 5$ (and therefore in particular for all infinite n) it will be shown that $A(N)$ is the only normal simple subgroup of G not reduced to the identity, and the matter will then be settled for all n . For the remainder of this section it is presumed that $n \geq 5$.

It is evident that $A(N)$ is normal, and it remains to prove that it is simple and is the only normal simple subgroup of G not reduced to the identity. (That $A(N)$ is simple is known [1] but for completeness we reproduce a proof here.) If X is any subset of the set N on which $\Sigma(N)$ acts, then the group of all permutations leaving fixed all elements outside X will be denoted by $\Sigma(X)$, and its alternating subgroup by $A(X)$. We shall consider only finite X with at least 5 elements; then $A(X)$ is simple. The union of $A(X_1)$ and $A(X_2)$ is contained in $A(X_1 \cup X_2)$, and $A(N)$ is the union of all the $A(X)$. If A' is a normal subgroup of $A(N)$ not reduced to the identity then it has a nontrivial intersection with some $A(X)$. This intersection, being a nontrivial normal subgroup of $A(X)$, must be all of $A(X)$, so $A' \supseteq A(X)$. Similarly A' contains $A(X')$ for all $X' \supseteq X$, so $A' \supseteq A(N)$ and $A(N)$ is therefore simple.

To show that $A(N)$ is the only normal simple subgroup of G not reduced to the identity it is sufficient now to show that if B is any other such subgroup then B has a nontrivial intersection with $A(N)$. Let b be any element of B other than the identity. It can not commute with every element of $A(N)$, whence there exists an a in $A(N)$ such that $c = aba^{-1}b^{-1} \neq 1$. Writing c as $(aba^{-1})b^{-1}$ it follows that c is in B since B was assumed normal. Similarly c is in $A(N)$. Therefore, $B \cap A(N)$ is not reduced to the identity, whence it follows that $B \supseteq A(N)$ and the proposition is proved: $A(N)$ is the only nontrivial normal simple subgroup of any subgroup G of $\Sigma(N)$ which contains $A(N)$. This in turn proves the proposition that $A(N)$ is construable as a structure derived from G . (It of course demonstrates that $A(N)$ is a characteristic subgroup of G .)

Characterization of a 3-cycle within $A(N)$. If N has fewer than 6 elements, then inside $A(N)$ a 3-cycle is completely characterized by the fact that it is an element of order 3. When N has 6 elements the characterization is known to be impossible. For the remainder of this section it will therefore be assumed that N has at least 7 elements, and for such an N a set of properties will be given which completely characterize a three-cycle within $A(N)$.

If G is a group and H a subgroup, then H will be said to be submaximal if there exists a proper subgroup of G properly containing H and if every such subgroup is maximal in G . For any element a in $A(N)$, let $C(a)$ denote the group of elements of $A(N)$ commuting with it. It will be shown that if a is a 3-cycle then $C(a)$ is a submaximal subgroup of $A(N)$, and is of order $(n-3)!3$ when n is finite (n =cardinality of N). Further, it will be shown that if a' is an element of $A(N)$ of order 3, if $C(a')$ is submaximal, and if for finite n , $C(a')$ is of order $(n-3)!3$, then a' is a 3-cycle.

By the carrier of a permutation in $\Sigma(N)$ will be meant the set of elements actually moved by the permutation. The elements of N will henceforth be called points. A permutation will be called finite or infinite according as the number of points in its carrier is finite or infinite. If X is any subset of N , the set of all permutations whose carriers are contained in X has been denoted by $\Sigma(X)$. The finite permutations in $\Sigma(X)$ form a smaller subgroup $\Sigma'(X)$ which in turn contains the alternating group $A(X)$ on X . The carrier of a subgroup G of $\Sigma(N)$ will be the union of the carriers of its elements. If G and H are subgroups of $\Sigma(N)$ whose carriers are disjoint, then we may form their internal direct product GH in $\Sigma(N)$. If X and Y are any two disjoint subsets of N then we may form in particular $\Sigma'(X)\Sigma'(Y)$, $[\Sigma'(X)\Sigma'(Y)] \cap A(N)$ (which will be denoted by $A(X, Y)$), and $A(X)A(Y)$. Each of these groups contains the next one mentioned, but the inclusions are not necessarily strict, since if Y contains but one point then $A(X, Y)$ and $A(X)A(Y)$ coincide, both being just $A(X)$. However, if X and Y each contain at least two points, then $A(X, Y)$ does strictly contain $A(X)A(Y)$, and the index of the latter group in the former is two. For an element $a = a_x a_y$ of $\Sigma'(X)\Sigma'(Y)$ can be in $A(N)$ if and only if either a_x and a_y are both even, in which case a is in $A(X)A(Y)$, or if a_x and a_y are both odd, and all elements of the second sort clearly lie in the same coset of $A(X)A(Y)$.

Under certain conditions a subgroup of $A(N)$ of the form $A(X)A(Y)$ is submaximal, a result for which the following lemma is needed. (It will be understood that a product ab is the effect of performing first b , then a .)

LEMMA. *Let X and Y be disjoint and $X \cup Y = N$, and suppose X and Y each have at least three elements. Let a be any element of $A(N)$ not in $A(X, Y)$ and suppose $a(X) \neq Y$. Then given any other element b in $A(N)$, there is an element c in the group G generated by a and $A(X)A(Y)$ such that bc is in $A(X, Y)$.*

PROOF. If b maps no element of X into Y , then b is already in

$A(X, Y)$ and c may be taken to be the identity. It is therefore sufficient to show that if b maps r elements of X into Y and r is positive, then an element c' can be found in G such that bc' maps at most $r-1$ elements of X into Y . To this end observe first that there can be found an element a' in G mapping exactly one element of X into Y (and hence also exactly one element of Y into X). For by the assumption on a , neither $Y \cap a^{-1}(X)$ nor $Y \cap a^{-1}(Y)$ is empty, and there is therefore an element d of $A(Y)$ carrying exactly one element of $a^{-1}(X)$ onto $a^{-1}(Y)$. One may then take $a' = ada^{-1}$. Suppose now that x_1 is a point of X carried into Y by b ; $b(x_1) = y_1$. There must also be a point of Y carried into X by b , $b(y'_1) = x'_1$. (It is immaterial that x'_1 may coincide with x_1 , or y'_1 with y_1 .) Let x'_2 be that point of X carried into Y by a' and y_2 that point of Y carried by it into X . If $a'(x'_2)$ were y'_1 and $a'(y_2)$ were x_1 , then a' would serve as the desired c' . Since the sets X and Y each have, by assumption, at least three elements, $A(X)$ and $A(Y)$ act transitively on them. There exists, therefore, an element a_x of $A(X)$ carrying $a'(y_2)$ into x_1 and an element a_y of $A(Y)$ carrying $a'(x'_2)$ into y'_1 . Choose then $c' = a_x a_y a'$, and the lemma is proved.

The foregoing lemma demonstrates in particular that $A(X, Y)$ is a maximal subgroup of $A(N)$. One may now derive the following:

LEMMA. *Let X and Y be disjoint, each contain at least three elements, and let $X \cup Y = N$. If either the cardinality of N is not finite, or if it is finite and the cardinality of X is different from that of Y , then $A(X)A(Y)$ is a submaximal subgroup of $A(N)$.*

PROOF. If N is infinite or if the cardinality of X is different from that of Y , then $a(X) \neq Y$ for any element a in N , so the hypothesis of the preceding lemma is satisfied. Suppose now that H, K are subgroups of N , $H \supset K \supset A(X)A(Y)$, and the inclusions are proper. Then it must be proved that $H = A(N)$. If $K = A(X, Y)$, then the matter is at an end since $A(X, Y)$ is maximal. Otherwise there exists an element a of K not in $A(X, Y)$. Let G denote the group generated by $A(X)A(Y)$ and a and let h be an element of H not in K . By the preceding lemma there exists an element c of G such that hc is an element d of $A(X, Y)$. The element d so obtained can not be in $A(X)A(Y)$ for h would then be $c^{-1}d$, an element of G and hence in K , against assumption. Since c is in H , it follows that H contains an element d of $A(X, Y)$ not in $A(X)A(Y)$, but since the index of $A(X)A(Y)$ in $A(X, Y)$ is two, it follows that H contains $A(X, Y)$. It does so properly, and since $A(X, Y)$ is maximal, $H = N$, which proves the lemma.

Suppose now that a is any 3-cycle in $A(N)$. Let X be the carrier of a , set $Y = N - X$, and let $C(a)$ be the group of all elements of $A(N)$ which commute with a . The set X has 3 elements, and since we are assuming throughout that there are at least seven points in N it follows that the cardinality of Y is at least 4 and hence in particular is different from that of X . Since $C(a) = A(X)A(Y)$, it follows from the preceding lemma that $C(a)$ is a submaximal subgroup of $A(N)$. The number of elements in $A(X)$ is 3 and that in $A(Y)$, if N is finite and has n elements, is $n-3$. There are then $(n-3)!3$ elements in $C(a)$.

It has been asserted that when $n \geq 7$, a 3-cycle a in $A(N)$ is completely characterized by the properties 1. a has order 3, 2. $C(a)$ is a submaximal subgroup of $A(N)$, and 3. if n is finite then $C(a)$ has order $(n-3)!3$. It having been demonstrated that a 3-cycle does in fact have these properties, it remains to show that if b is any element of order 3 in $A(N)$ such that properties 2. and 3. hold for b , then b is a 3-cycle. To this end, assume that b is an element of order 3 in $A(N)$ but not a 3-cycle. It will be shown that either property 2. or property 3. then fails to hold.

Being a finite permutation of order 3, b can be written as a product of a finite number of 3-cycles $b_1 \cdots b_r$ ($r > 1$), whose carriers $X_1, \cdots, X_r$, are disjoint. It will be shown first that if $X_1 \cup \cdots \cup X_r$ omits more than one point of N then $C(b)$ can not be submaximal in $A(N)$. Let the complement of $X_1 \cup \cdots \cup X_r$ be Y . If Y contains at least 2 points it is sufficient to show that $C(b)$ is not all of $A(X_1 \cup \cdots \cup X_r)A(Y)$, since it is certainly contained in that group which is itself not maximal. But no permutation in $C(b)$ can map any element of X_1 into X_2 unless it maps all of X_1 into X_2 , and there are certainly elements in $A(X_1 \cdots X_r)$ which do not have this property. This proves that if b is an element of $A(N)$ of order 3 but not a 3-cycle, then either the carrier of b is all of N , omits one point, or $C(b)$ is not submaximal. This settles the case when N is infinite. (In the case when N is infinite therefore, an element b of $A(N)$ of order three with the property that $C(b)$ is submaximal is indeed a 3-cycle.) When N is finite and $C(b)$ submaximal then $X_1 \cup \cdots \cup X_r$ must be all of N or omit exactly one point. The number of elements in $C(b)$ is then $r!3^r$, but if the number of elements n in N is greater than 7 this is smaller than $(n-3)!3$. Of the properties 1., 2., and 3., one has thus been shown to fail for a permutation b which is not a 3-cycle, and the characterization of a 3-cycle is therefore complete even in the finite case.

Reconstruction of N . In the first two sections of this paper it has

so far been demonstrated that if G is any subgroup of $\Sigma(N)$ which contains $A(N)$ then $A(N)$ may be considered as a structure derived from G , and the 3-cycles have furthermore been characterized within the structure of $A(N)$ except when the number n of points in N is 6. To any point p in N may be assigned the set $T(p)$ of all 3-cycles whose carriers contain p , and for $n > 3$ there is a canonical one-one correspondence between the points p of N and the sets $T(p)$. The reconstruction of N , and hence the proof of the main theorem, will be at an end when the subsets of the form $T(p)$ have been characterized within $A(N)$.

When $n = 4$ the complement of a set of the form $T(p)$ within the set of 3-cycles of $A(N)$ consists of a single 3-cycle and its inverse, and conversely, any 3-cycle and its inverse constitute the complement of a set of the form $T(p)$. The sets $T(p)$ are therefore completely characterized in the case when $n = 4$ and the reconstruction of N is finished for that case. We may therefore assume henceforth that $n > 4$.

For $n > 4$, in any set $T(p)$ there exist two 3-cycles whose carriers have exactly one point of N in common. Any other 3-cycle is in $T(p)$ if and only if it contains this point. To characterize $T(p)$ within $A(N)$ it is sufficient to express these assertions algebraically. The number of points which the carriers of two 3-cycles a and b have in common can be determined so: There are three points in common if and only if $a = b$ or $a = b^{-1}$, two points in common if and only if ab does not have order 5 and a and b do not commute, one point in common if and only if ab does have order 5, and no points in common if and only if a and b commute and are not identical or inverse to each other. Therefore, a statement about the number of points which the carriers of two 3-cycles have in common may henceforth be considered an algebraic statement.

Let p be any point of N and a, b be two 3-cycles in $T(p)$ whose carriers have exactly one point in common (namely p). The carrier of any other 3-cycle c in $T(p)$ must have at least one point in common with that of a and one point in common with that of b , but this is not sufficient for the carrier of c to contain p and therefore for c to be in $T(p)$. Let a be the permutation (p, q, r) and b be (p, s, t) . There are essentially two cases to consider. First, suppose that the carrier of c has exactly one point in common with that of a and exactly one point in common with that of b . If c were in $T(p)$ then it would be of the form (p, u, v) and if not in $T(p)$ it would be of the form, say, (q, s, w) (here w might be identical with u or v). One may distinguish algebraically between the cases by observing that in

the former the carrier of aca^{-1} is disjoint from that of b , but in the latter this does not hold. Second, suppose the carrier of c has more than one point in common with that of a . If the carrier of c has three points in common with that of a , or if it has two points in common with that of a and two points in common with that of b , then the carrier of c must contain p . We need then only investigate the case when the carrier of c has exactly two points in common with that of a and one in common with that of b . If c is in $T(p)$ then it must be of the form (p, q, s) , and if not, of the form, say, (q, r, s) . One may distinguish algebraically between the cases by observing that in the latter a is the conjugate of some power of c by some power of b , but in the former this is not the case.

When N has more than 4 elements, a set of the form $T(p)$ may therefore be fully characterized in the following fashion: 1. The set contains two 3-cycles a and b whose carriers have exactly one point in common. 2. If c is any 3-cycle in $A(N)$ then c is in the set if either a . Its carrier has exactly one point in common with that of a and one in common with that of b , and the carrier of aca^{-1} is disjoint from that of b , or b . Its carrier has more than one point in common with, say, that of a and at least one point in common with that of b , and a is not a conjugate of some power of c by some power of b . The characterization having also been given for $n=4$, the reconstruction of N from the 3-cycles of $A(N)$ is therefore accomplished for all N having 4 or more elements. Combining this with the fact that the 3-cycles have been characterized within $A(N)$ except when N has 6 elements, we have come to the end of the proof of the theorem asserted at the beginning of the paper:

THEOREM. *If G is any subgroup of $\Sigma(N)$ which contains $A(N)$ then every automorphism of G may be extended uniquely to an inner automorphism of $\Sigma(N)$ except when N has 3 or 6 elements.*

(For one may observe that the only cases not discussed, namely when $n=1$ or 2 , are trivial.)

Addenda to "Canonical constructions. I" George Whaples² has pointed out that when S has four elements the commutator of $(1, 2)$ is contained in the group of order 8 generated by $(1, 2)$, $(3, 4)$, and $(1, 2)(3, 4)$, hence is not maximal. In this case, therefore, one must characterize the transpositions differently in order that the proof of the theorem be complete. (The characterization given is valid for $n \geq 5, n \neq 6$). An elegant characterization of transpositions involving no

² The author wishes to thank both him and W. R. Scott for their comments.

counting arguments has been communicated to the author by W. R. Scott. A transposition x is an element of order two with the following properties: 1. There exists an element y of order 2 such that the order of $xy^{-1}xy$ is three. 2. For no element y of order two does the order of $xy^{-1}xy$ exceed three. This may be verified by direct computation (and settles the difficulty when $n=4$). Scott has also called the author's attention to the fact that the theorem that every automorphism of the finite symmetric group is inner ($n \neq 6$) was proved by J. König [4], and should properly be attributed to him.

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4. J. König, *Ueber rationale Funktionen von n Elementen und die allgemeine Theorie der algebraischen Gleichungen*, *Math. Ann.* vol. 14 (1879) esp. pp. 217–220.
5. J. Schreier and S. Ulam, *Über die Automorphismen der Permutationsgruppe der natürlichen Zahlenfolge*, *Fund. Math.* vol. 28 (1937) pp. 258–260. (The theorem that every automorphism of an infinite symmetric group is inner is announced here but proved only for the countable case.)

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