METRIZABILITY OF DECOMPOSITION SPACES

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1. Introduction. Let f be a quasicompact continuous mapping of a metric space S onto a topological space E ; that is, let f be a mapping of S onto E such that a subset Y of E is open (closed) in E if and only if $f^{-1}(Y)$ is open (closed) in S . As is well known,¹ E is then homeomorphic, in a natural way, to the hyperspace of the decomposition of S into the disjoint nonempty sets $F_p = f^{-1}(p)$ ($p \in E$), while conversely the hyperspace of every decomposition of S into disjoint nonempty sets F_p arises in this way, essentially uniquely. The object of this paper is to determine conditions under which E will be metrizable. Results of this nature have been known for some time, particularly for the special case in which f is closed (i.e., $f(X)$ is closed in E whenever X is closed in S)—or, equivalently, the case in which the decomposition $\{F_p\}$ is upper semi-continuous (i.e., if $F_p \subset U$ where U is open, there exists an open set V such that $F_p \subset V$ and every F_q meeting V is contained in U).² But it seems to have escaped notice that the problem has a simple complete answer in this case (Theorem 1), without any assumptions of compactness or separability.³ We shall also improve some known results about quasicompact images of locally compact spaces (Theorems 2 and 3), and obtain a criterion (Theorem 4) for the metrizability of E when f is open (i.e., when the decomposition is lower semi-continuous). Here the conditions we obtain are sufficient but not necessary; however, we show by examples that they are not superfluous. Finally we apply Theorem 1 to obtain a simple description of the case in which f is both open and closed (i.e., when the decomposition is continuous); it turns out that E is then always metrizable, with a quite convenient metric.

2. Closed mappings.

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¹ See [2, p. 65; 5; 10]. In the terminology of [2] "quasicompact continuous" = "stark-stetig." In the terminology of Bourbaki, E is the space S/R where R is the equivalence relation corresponding to f (i.e., $xRy \leftrightarrow f(x) = f(y)$).

² See [9, p. 123; 4]. The term "upper semi-continuous decomposition" has been used in the literature with several different meanings, but that given above seems to be generally adopted now. It is equivalent to "stetig Zerlegung" in [2], except that we do not require F_p to be closed.

³ Another, independent, proof of Theorem 1 has appeared after this paper was written; see K. Morita and S. Hanai, *Closed mappings and metric spaces*, Proc. Japan Acad. vol. 32 (1956) pp. 10–14.

THEOREM 1. *Let f be a closed continuous mapping of a metric space S onto a topological space E . Then the following statements are all equivalent:⁴*

- (i) E satisfies the first countability axiom.
- (ii) For each $p \in E$, $f^{-1}(p)$ has a compact frontier (in S).
- (iii) E is metrizable.

REMARK. E is in any case a T_1 space, for each $p \in E$ is of the form $f(x)$ where (x) is closed. Hence $F_p = f^{-1}(p)$ is closed in S , for each $p \in E$.

It is enough to prove (i) $\rightarrow$ (ii) and (ii) $\rightarrow$ (iii).

(i) $\rightarrow$ (ii). Let $\{W_n(p)\}$ ($n = 1, 2, \dots$) be a countable basis of open neighborhoods of $p \in E$. If $\text{Fr}(F_p)$ is not compact, there is a sequence $\{x_n\}$ ($n = 1, 2, \dots$) of points of $\text{Fr}(F_p)$ having no cluster point in $\text{Fr}(F_p)$, and consequently no cluster point in S . Now $f^{-1}(W_n(p))$ is an open set containing F_p ; as F_p is closed, $x_n \in f^{-1}(W_n(p))$, and so there exists $y_n \in S - F_p$ such that $y_n \in f^{-1}(W_n(p))$ and $\rho(x_n, y_n) < 1/n$, ρ denoting the distance in S . Let $Y = \{y_n\}$; Y is closed, since the sequence $\{y_n\}$ has no cluster point in S (else the sequence $\{x_n\}$ would). Hence $Q = f(Y)$ must be closed in E . By construction, $p \notin Q$; yet $p \in \bar{Q}$ since $W_n(p)$ meets Q in $f(y_n)$, and this contradicts the closedness of Q .

(ii) $\rightarrow$ (iii) The case in which the sets F_p are themselves compact has been established by S. Hanai,⁵ and the present proof uses the same idea as his. We construct for each $p \in E$ a sequence $\{W_n(p)\}$ ($n = 1, 2, \dots$) of open sets and prove (1) $W_1(p) \supset W_2(p) \supset \dots$, (2) $\bigcap_n W_n(p) = \{p\}$, (3) the sets $W_n(p)$ form a basis at p , (4) given a positive integer n and a point p of E , there exists m such that whenever $W_m(q)$ meets $W_n(p)$ we have $W_m(q) \subset W_n(p)$. By a theorem of Mrs. Frink [3], E will then be metrizable.

For any set $X \subset S$, write $S(X, \epsilon)$ to denote the open ϵ -neighborhood $\{y \mid \rho(y, X) < \epsilon\}$ of X . (If $X = \emptyset$, $S(X, \epsilon) = \emptyset$.) Now write $N_p^n = S(\text{Fr}(F_p), 1/n)$, $U_p^n = N_p^n \cup \text{Int}(F_p)$, $V_p^n = \bigcup \{F_q \mid F_q \subset U_p^n\}$, and $W_n(p) = f(V_p^n) = \{q \mid F_q \subset U_p^n\}$. Thus $W_n(p) = E - f(S - U_p^n)$; and, as U_p^n is open, it follows from the closedness of f that $W_n(p)$ and so V_p^n are open. Clearly $U_p^n \supset V_p^n \supset F_p$, so $W_n(p) \ni p$. The assertion (1) is trivial, and (2) follows from (3) since E is T_1 . To prove (3), let G be any open set (in E) containing p ; then $\text{Fr}(F_p) \subset F_p \subset f^{-1}(G)$, and so the closed sets $\text{Fr}(F_p)$, $S - f^{-1}(G)$, are disjoint. As $\text{Fr}(F_p)$ is com-

⁴ The implication (iii) $\rightarrow$ (ii) is due to Vainstein [7]. If S is separable, the implication (i) $\rightarrow$ (iii) is implicit in [10].

⁵ See [4]; another proof is implicit in [11, p. 367]. The general case can be deduced from the F_p -compact case; but, as the proof in [4] has a gap and that in [11] uses different basic definitions, it seems preferable to give a direct argument.

fact, the distance $d = \rho\{\text{Fr}(F_p), S - f^{-1}(G)\}$ is positive. Take $n > 1/d$; then $N_p^n \subset f^{-1}(G)$, and consequently $f^{-1}(G) \supset U_p^n \supset V_p^n$, so that $G \supset W_n(p)$, q.e.d.

To prove (4), suppose p and n given. If $\text{Int}(F_p) \neq \emptyset$, pick a point $x_p \in \text{Int}(F_p)$ arbitrarily. Now take m so large that

- (a) $m > 2n$,
- (b) if $\text{Fr}(F_p) \neq \emptyset$, $\rho(\text{Fr}(F_p), S - V_p^{2n}) > 2/m$,
- (c) if $\text{Int}(F_p) \neq \emptyset$, $S(x_p, 1/m) \subset F_p$.

Suppose $W_m(q)$ meets $W_m(p)$ in r , say, where we assume that $q \neq p$. We first show that $F_q \subset V_p^{2n}$. Take $w \in F_r \subset V_q^m \cap V_p^m \subset U_q^m \cap U_p^m$. If $w \in \text{Int}(F_q)$, then F_q meets V_p^m and consequently $F_q \subset V_p^m \subset V_p^{2n}$ as required. Since $w \in U_q^m = N_q^m \cup \text{Int}(F_q)$, we may assume now that $w \in N_q^m$. Again, if $w \in \text{Int}(F_p)$, then x_p exists; also F_p meets V_q^m and so $F_p \subset V_q^m$; hence $x_p \in V_q^m \subset \text{Int}(F_q) \cup N_q^m$, and as $x_p \notin F_q$ (for $p \neq q$) we have $x_p \in N_q^m$, which violates condition (c). Hence $w \in N_q^m \cap N_p^m$, and there exist points $x \in \text{Fr}(F_p)$, $y \in \text{Fr}(F_q)$, such that $\rho(w, x) < 1/m$ and $\rho(w, y) < 1/m$. It follows that $\rho(x, y) < 2/m$, and condition (b) shows that $y \in V_p^{2n}$. As F_q meets V_p^{2n} , we have $F_q \subset V_p^{2n}$.

Next we deduce $U_q^m \subset U_p^n$. The preceding argument has shown that $F_q \subset U_p^{2n} = N_p^{2n} \cup \text{Int}(F_p)$; hence $F_q \subset N_p^{2n}$. Let z be any point of U_q^m ; then there exists $y' \in F_q$ such that $\rho(z, y') < 1/m < 1/2n$, from (a). Since $y' \in N_p^{2n}$, there exists $x' \in \text{Fr}(F_p)$ such that $\rho(x', y') < 1/2n$. Thus $\rho(x', z) < 1/n$, so that $z \in N_p^n \subset U_p^n$. This proves $U_q^m \subset U_p^n$ if $p \neq q$; but if $p = q$ then $U_q^m \subset U_p^n$ trivially. Hence $V_q^m \subset V_p^n$ in any case, whence $W_m(q) \subset W_n(p)$, and the proof is complete.

COROLLARY 1. *If the conditions of the theorem are satisfied, and if S is separable, or locally separable, or compact, or locally compact, then E also has the corresponding property.*

Compactness and separability are preserved by arbitrary continuous mappings. Suppose S locally compact; we prove E locally compact at $p \in E$. Since $\text{Fr}(F_p)$ is compact, it is covered by a finite number of open sets with compact closures. Thus, if n is large enough, N_p^n has a compact closure. So therefore have $V_p^n \cap N_p^n$ and (since f is closed) $f(V_p^n \cap N_p^n)$. But, if $\text{Fr}(F_p) \neq \emptyset$, $f(V_p^n \cap N_p^n) = f(V_p^n) = W_n(p)$. If $\text{Fr}(F_p) = \emptyset$, F_p is an open inverse set, so $\{p\}$ is a compact neighborhood of p . Finally, the proof for local separability is entirely similar.

REMARK. Corollary 1 does not apply to local peripheral compactness.⁶

COROLLARY 2. *If f is a closed continuous mapping of a metric space*

⁶ A space is locally peripherally compact if every point has arbitrarily small neighborhoods with compact frontiers. Every 0-dimensional space is locally peripherally compact.

S onto a locally countably compact space E , then E is metrizable (and consequently the sets $\text{Fr } (f^{-1}(p))$ are compact).

For each point p of E has a sequence of open neighborhoods $X_n(p) = \{q \mid F_q \subset S(F_p, 1/n)\}$ with the property $\bigcap_n X_n(p) = \{p\}$. Since E is locally countably compact and regular,⁷ one easily derives open neighborhoods $Y_n(p)$ of p such that $\text{Cl}(Y_n(p)) \subset X_n(p)$, $Y_n(p) \supset Y_{n+1}(p)$, and $\text{Cl}(Y_1(p))$ is countably compact; here Cl denotes the closure. It is a straightforward matter to verify that the sets $Y_n(p)$ form a basis of neighborhoods at p ; hence E satisfies the first axiom of countability, and Theorem 1 applies.

3. Monotone mappings. The condition that f be *closed* cannot be omitted in Theorem 1, in general [2, p. 70, Example 2], even if f is open, S a subset of the plane, E a Hausdorff space with a countable basis, and the sets $f^{-1}(p)$ are all compact. However, in some cases the closedness of f follows from the other hypotheses. This is so, for example, if S is compact and E is Hausdorff. A less obvious example is the following, which generalizes a theorem of A. V. Martin [5, Theorem 5].

THEOREM 2. *Let E be the hyperspace of a decomposition of a locally peripherally compact metric space S into connected sets with compact frontiers; equivalently, let $E = f(S)$ where f is a quasicompact continuous mapping such that, for each $p \in E$, $f^{-1}(p)$ is connected and $\text{Fr } (f^{-1}(p))$ is compact. Then, if E is a Hausdorff space, f is closed (and consequently E is metrizable, by Theorem 1); further, E is then locally peripherally compact.*

As before, we write $F_p = f^{-1}(p)$; we first prove that the decomposition $\{F_p\}$ of S is upper semi-continuous. Given any open set $U \supset F_p$, cover $\text{Fr } (F_p)$ by a finite number of open sets $U_1, U_2, \dots, U_m$ such that $U_i \subset U$ and $\text{Fr } (U_i)$ is compact ($1 \leq i \leq m$). Let $V = U_1 \cup U_2 \cup \dots \cup U_m \cup \text{Int } (F_p)$; then $F_p \subset V \subset U$, and the set $\text{Fr } (V)$, being a closed subset of $\text{Fr } (U_1) \cup \dots \cup \text{Fr } (U_m) \cup \text{Fr } (F_p)$, is compact. Take $W_n = S(\text{Fr } (F_p), 1/n) \cup \text{Int } (F_p)$, an open set containing F_p ; it is enough to prove that, for some n , every F_q meeting W_n is contained in V . Suppose this false; then, for each n , we obtain $q_n \in E$ such that F_{q_n} meets both W_n and $S - V$. Since $q_n \neq p$ (for $F_p \subset V$), F_{q_n} contains a point y_n of $S(\text{Fr } (F_p), 1/n)$, and there exists $x_n \in \text{Fr } (F_p)$ such that $\rho(x_n, y_n) < 1/n$. Again, for all n greater than some n_0 , we have $1/n < \rho(\text{Fr } (F_p), S - V)$, so that $W_n \subset V$. Hence F_{q_n} meets both V and $S - V$, and so meets $\text{Fr } (V)$, say in z_n ($n > n_0$).

For a suitable subsequence of values n' of n , we have $z_{n'} \rightarrow z \in \text{Fr } (V)$,

⁷ In fact, any closed continuous image of a normal space is normal [10].

$x_{n'} \rightarrow x \in \text{Fr}(F_p)$, and therefore $y_{n'} \rightarrow x$. Now $z \notin F_p$, since V contains F_p and is disjoint from $\text{Fr}(V)$. Let $f(z) = q$; then $q \in E - (p)$ and $z \in F_q$. Since E is a Hausdorff space, there exist disjoint open inverse sets $Y \supset F_p$, $Z \supset F_q$. If n' is large enough, $y_{n'} \in Y$ and $z_{n'} \in Z$; thus $F_{q_{n'}}$ meets, and so is contained in, both Y and Z , giving a contradiction.

This proves the decomposition upper semi-continuous; the mapping f is therefore closed, and Theorem 1 applies. Finally, to prove E locally peripherally compact, suppose $p \in E$ and an open set $G \ni p$ given; as above, we construct an open set $V \supset F_p$ such that $V \subset f^{-1}(G)$ and $\text{Fr}(V)$ is compact. Let $W = \{q \mid F_q \subset V\} = E - f(S - V)$, an open set such that $p \in W \subset G$. Write $X = f^{-1}(W)$; clearly $X \subset V$. Suppose q is any point of $\text{Fr}(W)$; then $q \notin W$, so F_q meets $S - V$. But $q \in \overline{W} = \text{Cl}(f(X)) = f(\overline{X})$ (since f is closed), so $q \in f(\overline{V})$ and F_q also meets $\overline{V}$. Being connected, F_q meets $\text{Fr}(V)$; hence $q \in f(\text{Fr}(V))$. This proves $\text{Fr}(W) \subset f(\text{Fr}(V))$, which is compact; thus $\text{Fr}(W)$ is also compact, q.e.d.

4. General quasicompact mappings. The condition that the sets F_p are all connected cannot be omitted from Theorem 2, even if they are all compact and S is a locally compact subset of the plane; examples to show this are easily constructed. We have, however, the following theorem (which generalizes another theorem of Martin [5, Corollary to Theorem 4]), in which the mapping f can in fact be neither open nor closed.

THEOREM 3. *If f is a quasicompact continuous mapping of a locally compact separable metric space S onto a Hausdorff space E with a locally countable basis, then E is a locally compact separable metric space.*

LEMMA 1. *Let f be a quasicompact continuous mapping of a topological space S onto a Hausdorff space E with a locally countable basis at $p \in E$, and let $\{U_n\}$ ($n = 1, 2, \dots$) be an increasing sequence of open sets of S such that $\bigcup U_n \supset F_p = f^{-1}(p)$. Then, for some n , $p \in \text{Int}(f(U_n))$.*

Let $\{W_n\}$ ($n = 1, 2, \dots$) be a basis of neighborhoods of p ; we may suppose $W_1 \supset W_2 \supset \dots$. Without loss of generality, we may assume that each U_n meets F_p . We show that, for some n , $f(U_n) \supset W_n$. Suppose not; then, for each n , there is a point $q_n \in W_n - f(U_n)$. Write $Q = \{q_n\}$ ($n = 1, 2, \dots$), $X = f^{-1}(Q)$; as $p \in \overline{Q} - Q$, Q is not closed, and therefore X is not closed and there exists a point $x \in \overline{X} - X$. Then $f(x) \in \overline{Q} - Q$, and it is easy to deduce (from the fact that E is Hausdorff and $q_n \rightarrow p$) that $f(x) = p$. Hence $x \in F_p$, so $x \in U_n$ for some n . Then $U_n - \bigcup \{f^{-1}(q_m), 1 \leq m \leq n\}$ is an open set containing x but

disjoint from X , contradicting $x \in \overline{X}$ and establishing the lemma.

LEMMA 2. *If E is the hyperspace of a decomposition $\{F_p\}$ of a locally compact topological space S , for which each $\text{Fr}(F_p)$ is σ -compact,⁸ and if E is Hausdorff and has a locally countable base, then E is locally compact.*

As usual, we use f to denote the natural quasicompact mapping of S onto E , so that $f^{-1}(p) = F_p$. By hypothesis, given $p \in E$, we can write $\text{Fr}(F_p) = \bigcup K_n$ ($n = 1, 2, \dots$) where K_n is compact. Since S is locally compact, K_n can be covered by finitely many open sets with compact closures; in this way we obtain open sets $G_n \supset K_n$ such that $\text{Cl}(G_n)$ is compact and $G_1 \subset G_2 \subset \dots$. Applying Lemma 1 to $U_n = G_n \cup \text{Int}(F_p)$, we have $p \in \text{Int}(f(U_n))$ for some n . If $\text{Fr}(F_p) \neq \emptyset$, this gives $p \in \text{Int}(f(G_n)) \subset f(\text{Cl}(G_n))$, which is compact and closed in the Hausdorff space E . If $\text{Fr}(F_p) = \emptyset$, $\{p\}$ is a compact neighborhood of p .

LEMMA 3. *If E is the hyperspace of a decomposition $\{F_p\}$ of a topological space S with a countable base, and if E is Hausdorff and has a locally countable base, then E has a countable base.*

(For upper semi-continuous decompositions, this was proved by Whyburn [10, Theorem 10].)

Let $\mathcal{B} = \{B_m\}$ be a countable base of open sets of S ; we prove E has a base of the form $\{\text{Int}(f(U))\}$ where U is a finite union of sets of $\mathcal{B}$. Given any $p \in E$ and any open set $G \ni p$, we have $F_p \subset f^{-1}(G)$, and can cover F_p by a sequence of sets $B_{m_1}, B_{m_2}, \dots$, of $\mathcal{B}$, all satisfying $B_{m_i} \subset f^{-1}(G)$. Write $U_n = B_{m_1} \cup \dots \cup B_{m_n}$; by Lemma 1 we have $p \in \text{Int}(f(U_n))$ for some n , where $f(U_n) \subset G$, and the lemma is proved.

PROOF OF THEOREM 3. As S is here separable metric and locally compact, the same is true of each $\text{Fr}(F_p)$, which is therefore σ -compact. By Lemma 2, E is locally compact, and therefore regular. Also E has a countable base, by Lemma 3, and so it is metrizable.

As another immediate consequence of Lemma 3, we have:

COROLLARY. *If f is a quasicompact continuous mapping of a separable metric space S onto a regular space E with a locally countable base, then E is separable metric.*

The extension of Theorem 3 (and its corollary) to *nonseparable* spaces S presents unexpected difficulties. We shall see from examples (§6) that even an *open* continuous image of a locally compact metric space need not be metrizable, even if it is regular (and has a locally countable base) and the sets $\text{Fr}(F_p)$ are compact. One fairly im-

⁸ A set is σ -compact if it is the union of countably many compact sets.

mediate extension of Theorem 3 is as follows.

THEOREM 3'. *Let f be a quasicompact continuous mapping of a locally compact metric space S onto a Hausdorff space E with a locally countable base, and suppose that the sets $\text{Fr } (f^{-1}(p))$ are separable and that E is paracompact. Then E is a locally compact metric space.*

For the argument proving Theorem 3 here shows that each $p \in E$ has a compact neighborhood of the form $f(\bar{G})$ where $\bar{G}$ is compact. Since E is Hausdorff, the mapping f restricted to $\bar{G}$ is closed; hence, by a well-known special case of Theorem 1, $f(\bar{G})$ is metrizable. Thus E is locally metrizable. Since E is paracompact, E has a locally finite covering by closed metrizable sets, and is therefore [6] metrizable.

The condition in Theorem 3' that E be paracompact cannot be dispensed with, in general, even if f is open (§6). I do not know whether it is superfluous if the sets $f^{-1}(p)$ are assumed to be separable. If f is open and the sets $f^{-1}(p)$ are separable, the paracompactness of E follows from the other hypotheses (see the corollary to Theorem 4 below).

5. Open mappings. If f is open, Theorem 3 has the following extension to not necessarily separable spaces; perhaps the most interesting feature of the theorem is that none of the hypotheses is superfluous (§6).

THEOREM 4. *If f is an open continuous mapping of a metric, locally separable space S onto a regular space E , and if for each $p \in E$ the set $f^{-1}(p)$ is separable, then E is metrizable and locally separable.*

LEMMA 1. *The theorem is true if S is separable, and E is then separable.*

This is a special case of the corollary to Theorem 3; it is also obvious directly.

LEMMA 2. *If $f(S) = E$, where f is open and continuous and each $f^{-1}(p)$ is separable, then for every separable subset Y of E , $f^{-1}(Y)$ is separable.*

Let $Q = \{q_m\}$ ($m = 1, 2, \dots$) be a countable subset of Y such that $\bar{Q} \supset Y$, and for each m let X_m be a countable dense subset of $f^{-1}(q_m)$. Write $X = \bigcup X_m$; it is easy to see that the countable set X is dense in $f^{-1}(Y)$.

PROOF OF THEOREM 4. By a theorem of Alexandroff [1], S can be expressed as a union of pairwise disjoint nonempty open sets S_λ , each of which is separable. Write $S_\lambda \sim S_\mu$ to mean that there exists a finite sequence $\lambda = \lambda_0, \lambda_1, \dots, \lambda_k = \mu$ such that each set $f(S_{\lambda_{i-1}})$ meets

$f(S_{\lambda_i})$ ($i=1, 2, \dots, k$). It is easily verified that $\sim$ is an equivalence relation. Let the union of the S_μ 's equivalent to S_λ be T_λ ; thus T_λ is open, and T_λ and $T_{\lambda'}$ are either identical or disjoint. Further, T_λ is an inverse set, (i.e., equals $f^{-1}(f(T_\lambda))$). For suppose $F_p = f^{-1}(p)$ meets T_λ , say in $y \in F_p \cap S_\mu$ where $S_\mu \sim S_\lambda$. If x is any point of F_p , we have $x \in \text{some } S_\nu$ and $p \in f(S_\nu) \cap f(S_\mu)$; hence $S_\nu \sim S_\mu \sim S_\lambda$, so that $x \in T_\lambda$. Thus $F_p \subset T_\lambda$ whenever F_p meets T_λ , and T_λ is an inverse set. It follows that the distinct sets $f(T_\lambda)$ are disjoint and open, and they cover E ; to prove the theorem, it will suffice to prove that each $f(T_\lambda)$ is separable metric, and by Lemma 1 it suffices to prove that each T_λ is separable.

Now let T_λ^n denote the union of those sets S_μ which can be reached from S_λ in at most n steps—that is, for which there is a sequence $\lambda = \lambda_0, \lambda_1, \dots, \lambda_k = \mu$, of the type used to define $\sim$, with $k \leq n$. Clearly $T_\lambda = T_\lambda^0 \cup T_\lambda^1 \cup \dots$, and it is enough to prove that each T_λ^n is separable. Suppose this true for one particular value of n . Then T_λ^{n+1} consists of T_λ^n together with those sets S_μ for which $f(S_\mu)$ meets $f(T_\lambda^n)$ —that is, for which S_μ meets $f^{-1}(f(T_\lambda^n))$. By hypothesis, T_λ^n is separable; hence so is $f(T_\lambda^n)$, and Lemma 2 now shows that $f^{-1}(f(T_\lambda^n))$ is also separable. It can therefore meet at most countably many of the disjoint open sets S_μ . Thus T_λ^{n+1} is a union of countably many separable sets, and is again separable. Since $T_\lambda^0 = S_\lambda$ and is separable, the separability of T_λ^n follows for all n ; and the theorem is proved.

COROLLARY. *If f is an open continuous mapping of a locally compact metric space S onto a Hausdorff space E , and if for each $p \in E$ the set $f^{-1}(p)$ is separable, then E is a locally compact metric space. (Compare Theorem 3'.)*

For, from the second lemma to Theorem 3, E is locally compact, and therefore regular; now Theorem 4 applies.

6. Counterexamples. The following examples show the need for the restrictions imposed in Theorems 3, 3' and 4.⁹ In each case, f will be an *open* (and so certainly quasi-compact) continuous mapping of a metric space S onto a nonmetrizable space E ; the openness of f is easily verified from the fact that E is in each case the hyperspace of a decomposition $\{F_p\}$ of S which is lower semi-continuous (that is, given any open set U meeting F_p , there exists an open set $V \supset F_p$ such that every F_q meeting V meets U). In the first two examples, S is locally compact; in all three, the openness of f guarantees that

⁹ See [2, p. 70, Ex. 1, 2] for examples showing that "Hausdorff" cannot be replaced by " T_1 " in Theorems 3 and 3' (even if S is compact), and that "regular" cannot be replaced by "Hausdorff" in Theorem 4 (even if S is separable).

E will have a locally countable base.

(1) For each countable ordinal α , let E_α denote the set of ordinals not exceeding α ; in the usual topology, E_α is a compact metric space, and we suppose it metrized with diameter <1 , say with metric ρ_α . Let S be the set of all ordered pairs (α, β) , where α, β are countable ordinals and $\alpha \geq \beta$. Define a distance ρ on S by: $\rho\{(\alpha, \beta), (\alpha', \beta')\} = \rho_\alpha(\beta, \beta')$ if $\alpha = \alpha'$, 1 otherwise. Thus S is the discrete union of the spaces E_α , and is locally compact (all sets of diameter less than 1 having compact closures). The mapping f given by $f(\alpha, \beta) = \beta$ is an open continuous mapping of S onto the space E of all countable ordinals (in its usual topology); the corresponding decomposition is $\{F_\beta\}$ where $F_\beta = \{(\alpha, \beta) | \alpha \geq \beta\}$. Here E is a completely normal Hausdorff space, and is locally compact and has a locally countable base, but is not metrizable.

(2) Our second example is similar, but E will satisfy fewer separation axioms; to compensate for this, the sets F_p will have compact frontiers. Let I denote the unit interval $0 \leq x \leq 1$, and let A, R denote respectively the sets of irrational and rational numbers in I . For each $a \in A$, choose *one* sequence $\{r_n(a)\} \rightarrow a$ with $r_n(a) \in R$ (e.g., by using the decimal expansion). Now take $S = (R \times A) \cup A$ metrized as follows: $\rho\{(r_m(a), a), a\} = 1/m$, $\rho\{(r_m(a), a), (r_n(a), a)\} = |1/m - 1/n|$, and all other distances between distinct points are 1. Thus S is the discrete union of sets $S_a = (R \times a) \cup \{a\}$, and S_a consists of a sequence $\{(r_m(a), a)\}$ converging to a , together with a discrete sequence of points (the other points (r, a)). Clearly S is a locally compact metric space (all closed sets of diameter less than 1 being compact). Now map S in I by: $f((r, a)) = r$, $f(a) = a$. The corresponding decomposition is characterized by $F_r = \{(r, a) | a \in A\}$, $F_a = \{a\}$. Each F_r ($r \in R$) is both open and closed in S , so in any case $\text{Fr}(F_p)$ is compact ($p \in I$) and has, in fact, at most one point. Further, from the fact that each F_p is either open or consists of a single point, it is immediate that the decomposition $\{F_p\}$ is lower semi-continuous; thus f is an open continuous mapping of S onto the hyperspace E of the decomposition, which is easily seen to consist of I with the following topology. A typical neighborhood of $r \in R$ is (r) ; a typical neighborhood of $a \in A$ consists of a together with all points $r_n(a)$ for n sufficiently large. These neighborhoods are closed, as well as open, in E ; hence E is a regular Hausdorff space. But E is separable (R is a countable dense set), yet contains an uncountable discrete set (A); hence E is not metrizable. (If the continuum hypothesis is true, E cannot be normal; see F. B. Jones, *Concerning normal and completely normal spaces*, Bull. Amer. Math. Soc. vol. 43 (1937) pp. 671-677.)

(3) In our last example, the sets F_p will themselves be compact,

though S will no longer be locally separable. Let N = set of all positive integers, G = set of all mappings g of N in N . Roughly, S will consist of N , G , a sequence of points converging to each $g \in G$, and c sequences of points (one for each $g \in G$) converging to each $n \in N$. Precisely, we take $S = G \cup (G \times N) \cup N \cup (N \times G \times N)$, and define a metric ρ on S as follows:

$$\begin{aligned} \rho\{g, (g, n)\} &= \frac{1}{n}; & \rho\{(g, n), (g, m)\} &= \frac{1}{n} + \frac{1}{m} & \text{if } n \neq m; \\ \rho\{n, (n, g, m)\} &= \frac{1}{m}; & \rho\{(n, g, m), (n, g', m')\} &= \frac{1}{m} + \frac{1}{m'} \\ & & & \text{if } (g, m) \neq (g', m'); \end{aligned}$$

and all other distances between distinct points are 2. It is easily verified that ρ is a metric. Note that the points (g, m) and (n, g, m) are isolated, each having distance at least $1/m$ from any other point. Decompose S into the 2-point sets $F_{gn} = \{(g, n), (n, g, g(n))\}$ and the other single points. The decomposition is lower semi-continuous (for the sets F_p with more than one point are open), and so corresponds to an open continuous mapping f of S onto the hyperspace E ; further the sets $F_p (= f^{-1}(p))$ are all compact (having, in fact, at most 2 points). It is a straightforward matter to verify that E is a regular T_1 space. Now the sets $f(G), f(N)$ are disjoint and closed in E ; if E were normal, there would be disjoint open sets $U \supset f(N), V \supset f(G)$. Then $N \subset f^{-1}(U)$, which is open in S ; hence, for each $n \in N$, we can choose $m = h(n)$ (say) large enough for $S(n, 1/(m-1)) \subset f^{-1}(U)$, and this gives $f(n, g, h(n)) \in U$. This is true for all $g \in G$, and so in particular for $g = h$; thus $f(h, n) = f(n, h, h(n)) \in U$, for all $n \in N$. But if n is large enough, $(h, n) \in S(h, 1/(n-1)) \subset f^{-1}(V)$, so that $f(h, n) \in V$, contradicting $U \cap V = \emptyset$. Thus E is not normal, and so certainly not metrizable.¹⁰

7. Open-closed mappings. We conclude by considering the case in which the mapping f of the metric space S is simultaneously open and closed—or, equivalently, in which the decomposition $\{F_p\}$ of S is continuous (i.e., both upper and lower semi-continuous). The essential content of the following theorem is due to Wallace [8], but Theorem 1 makes the proof very simple.

¹⁰ Incidentally, example (3) answers a question raised in [2, p. 70] by giving a decomposition of a normal space for which the hyperspace is regular but not normal. Another example answering this question has been given by T. Ganea, *On the Prüfer manifold and a problem of Alexandroff and Hopf*, Acta Scientiarum Math. vol. 15 (1954) pp. 231–235.

THEOREM 5. *Necessary and sufficient conditions for a decomposition of a metric space S into disjoint nonempty closed sets F_p to be continuous are:*

- (a) *Each F_p is either open or compact.*
- (b) *Given a compact F_p and $\epsilon > 0$, there exists $\delta > 0$ such that whenever $\rho(F_p, F_q) < \delta$ we have $d(F_p, F_q) < \epsilon$ (d denoting the Hausdorff distance).*

To prove (a) and (b) necessary, we note that the decomposition space E now has a locally countable basis (for the corresponding mapping f of S onto E is open), so by Theorem 1 each $\text{Fr}(F_p)$ is compact. If $\text{Int}(F_p) = \emptyset$, this proves F_p compact. If $\text{Int}(F_p) \neq \emptyset$, $f(\text{Int}(F_p)) = \{p\}$ is open in E , so F_p is open in S . Thus (a) follows; and (b) is now a routine restatement of upper and lower semi-continuity. Conversely, if (a) and (b) are given, it is easy to deduce that the decomposition is both upper and lower semi-continuous.

COROLLARY.¹¹ *If $f(S) = E$, where f is an open, closed, continuous mapping of a metric space S , then E is metrizable, with metric σ given by $\sigma(p, q) = d(f^{-1}(p), f^{-1}(q))$.*

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¹¹ This corollary is due to V. K. Balanchandran, *A mapping theorem for metric spaces*, Duke Math. J. vol. 22 (1955) pp. 461–464.