TWO NOTES ON FORMAL POWER SERIES

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This paper consists of two more or less disjoint notes, the first on integral formal power series in several variables and the second concerning the generalized Puiseux expansion of a certain algebraic function of one variable over a modular field.

1. **Analytically independent formal (integral) power series.** Let k be an arbitrary field and let L_n be the ring of formal series $k[[x_1, x_2, \cdots, x_n]]$ in n variables $x_1, x_2, \cdots, x_n$ with coefficients in k. We recall that given m elements $f_i(x_1, x_2, \cdots, x_n)$, $i=1, 2, \cdots, m$, of L_n one says that $f_1, f_2, \cdots, f_m$ are analytically dependent if there exists $0 \neq H(Z_1, Z_2, \cdots, Z_m) \in k[[Z_1, Z_2, \cdots, Z_m]]$ such that $H(f_1(x_1, x_2, \cdots, x_n), \cdots, f_m(x_1, x_2, \cdots, x_n))=0$. If $f_1, f_2, \cdots, f_m$ are not analytically dependent then they are said to be analytically independent. Given an infinite number $f_1, f_2, \cdots$ of elements of L_n, these elements are said to be analytically independent if every finite number of them are analytically independent. Professor Samuel asked me whether L_2 contains three analytically independent elements. The answer is given in the following

PROPOSITION. *If $n>1$ then L_n contains an infinite number of analytically independent elements.*

PROOF. It is known that there exists an infinite number $g_1(y)$, $g_2(y), \cdots$ of formal power series in one variable y with coefficients in k which are algebraically independent over k (Lemma 1 of [4]). Let $f_i(x_1, x_2, \cdots, x_n) = x_2 g_i(x_1)$ for $i=1, 2, \cdots$. Let $H(Z_1, Z_2, \cdots, Z_m)$ be an element of $k[[Z_1, Z_2, \cdots, Z_m]]$ for which $H(f_1(x_1, x_2, \cdots, x_n)$, $f_2(x_1, x_2, \cdots, x_n), \cdots, f_m(x_1, x_2, \cdots, x_n))=0$. Let $H(Z_1, Z_2, \cdots, Z_m) = \sum_{j=0}^{\infty} H_j(Z_1, Z_2, \cdots, Z_m)$ where $H_j(Z_1, Z_2, \cdots, Z_m)$ is a form of degree j in $k[Z_1, Z_2, \cdots, Z_m]$. Then

$$0 = H(f_1(x_1, x_2, \cdots, x_n), \cdots, f_m(x_1, x_2, \cdots, x_n))$$

$$= \sum_{j=0}^{\infty} x_2^j H_j(g_1(x_1), g_2(x_1), \cdots, g_m(x_1)).$$

Therefore

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$$H_j(g_1(x_1), g_2(x_1), \cdots, g_m(x_1)) = \text{the coefficient of } x_2^j \text{ in } H(f_1, \cdots, f_m) \\ = 0 \quad \text{for} \quad j = 1, 2, \cdots.$$

Since $g_1(x_1), g_2(x_1), \cdots, g_m(x_1)$ are algebraically independent over k, we must have $H_j(Z_1, Z_2, \cdots, Z_m) = 0$ for $j = 1, 2, \cdots$, i.e., $H(Z_1, Z_2, \cdots, Z_m) = 0$. Therefore $f_1(x_1, x_2, \cdots, x_n), f_2(x_1, x_2, \cdots, x_n), \cdots$ are analytically independent.

Let us remark that any two elements $f(x_1)$ and $g(x_1)$ of L_1 are analytically dependent. If either f or g is zero then there is nothing to prove and hence we may suppose that $f(x_1) \neq 0 \neq g(x_1)$. First assume that at least one of the two elements $f(x_1)$ and $g(x_1)$ is a non-unit; say $f(x_1)$ is a non-unit. Then by Proposition 3.5 of Chevalley [2], L_1 is a finite module over $k[[f(x_1)]]$ and hence $g(x_1)$ is integral over $k[[f(x_1)]]$. Therefore there exists $0 \neq H(X, Y) \in k[[X]][Y]$ with $H(f(x_1), g(x_1)) = 0$. Now assume that $f(x_1)$ and $g(x_1)$ are both units. Then $f^*(x_1) = f(x_1) - ag(x_1)$ is a non-unit, where $a = f(0)/g(0)$. Hence by the previous case, there exists $0 \neq H(X, Y) \in k[[X, Y]]$ with $H(f^*(x_1), g(x_1)) = 0$. Let $H^*(X, Y) = H(X - aY, Y)$. Then $H^*(f(x_1), g(x_1)) = 0$. Also $H^*(X, Y) \neq 0$ since $X \rightarrow X - aY$, $Y \rightarrow Y$ is an automorphism of L_1.

2. **A fractional power series.** Let k be an algebraically closed field of characteristic p. If $p = 0$ then the theorem of Puiseux expansion is valid, i.e., any polynomial $F(Y) = y^n + f_1(X)Y^{n-1} + \cdots + f_n(X)$ with $f_i(X) \in k(X)$ can be factored in the form

$$F(Y) = \prod_{i=1}^{n} (Y - g_i(X^{1/m})),$$

where m is some positive integer and where $g_1(X), g_2(X), \cdots, g_n(X)$ are in the quotient field of $k[[X]]$, i.e., in the integral formal power series field $k((X))$. If $p \neq 0$ then such a factorization is not always possible. A typical example of this is given by Chevalley on p. 64 of [3], namely:

$$F(Z) = Z^p - Z - X^{-1}.$$

If we force a factorization, we get the following generalized Puiseux expansion (where the denominators of the indices of X are unbounded):

$$F(Z) = \prod_{i=0}^{p-1} \left(Z + i - \sum_{i=1}^{\infty} X^{-1/p^i} \right).$$

Or getting rid of the poles, we get alternatively:

$$Z^p - X^{p-1}Z - 1 = \prod_{i=0}^{p-1}\left(Z + iX - \sum_{i=0}^{\infty} X^{1-p^{-i}}\right).$$

These factorizations can be verified directly. They were used in discovering some of the examples discussed in [1].

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ON THE COMPOSITUM OF ALGEBRAICALLY CLOSED SUBFIELDS

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Professor Igusa asked me the following question: Given a field K, is the compositum of all the (absolutely) algebraically closed subfields of K itself algebraically closed (of course, we are assuming that this compositum is not empty, i.e., that K contains the algebraic closure of its prime field)? We shall show in §1 that the *answer* to this question *is in the negative in general.* In §2, we shall give a special case in which the answer is in the affirmative.

1. **The algebraic closures of $k(x)$, $k(y)$ and $k(x, y)$.** Let k be an arbitrary algebraically closed field, $L = k(x, y)$ where x and y are algebraically independent over k, L^* = an algebraic closure of L, $M = k(x)$, $N = k(y)$, M^* = the algebraic closure of M in L^*, N^* = the algebraic closure of N in L^*, and K = the compositum of L^* and M^*. Let T be the compositum of all the algebraically closed subfields of K. Then $M^* \subset T$ and $N^* \subset T$ and hence $T = K$. *We shall prove below that K cannot be algebraically closed.* In fact we shall show that in some sense K is much nearer to L than it is to L^* and hence that K is far from being algebraically closed.

Embed $L = k(x, y)$ canonically into the formal power series field

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