$$Z^p - X^{p-1}Z - 1 = \prod_{i=0}^{p-1}\left(Z + iX - \sum_{i=0}^{\infty} X^{1-p^{-i}}\right).$$

These factorizations can be verified directly. They were used in discovering some of the examples discussed in [1].

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ON THE COMPOSITUM OF ALGEBRAICALLY CLOSED SUBFIELDS

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Professor Igusa asked me the following question: Given a field K, is the compositum of all the (absolutely) algebraically closed subfields of K itself algebraically closed (of course, we are assuming that this compositum is not empty, i.e., that K contains the algebraic closure of its prime field)? We shall show in §1 that the *answer* to this question *is in the negative in general.* In §2, we shall give a special case in which the answer is in the affirmative.

1. The algebraic closures of $k(x)$, $k(y)$ and $k(x, y)$. Let k be an arbitrary algebraically closed field, $L=k(x, y)$ where x and y are algebraically independent over k, L^* = an algebraic closure of L, $M=k(x)$, $N=k(y)$, M^* = the algebraic closure of M in L^*, N^* = the algebraic closure of N in L^*, and K = the compositum of L^* and M^*. Let T be the compositum of all the algebraically closed subfields of K. Then $M^*\subset T$ and $N^*\subset T$ and hence $T=K$. *We shall prove below that K cannot be algebraically closed.* In fact we shall show that in some sense K is much nearer to L than it is to L^* and hence that K is far from being algebraically closed.

Embed $L=k(x, y)$ canonically into the formal power series field

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$L_1=k((x, y))$. Let L_1^* be an algebraic closure of L_1. Then we may assume that L^* is the algebraic closure of L in L_1^*. Let $M_1=k((x))\subset L_1$, $N_1=k((y))\subset L_1$, and let M_1^* and N_1^* be the algebraic closures respectively of M_1 and N_1 in L_1^*. Let K_1 be the compositum of M_1^* and N_1^*. We thus have: $k(x, y)=L\subset K\subset L^*\subset L_1^*$ and $K\subset K_1$. Let p be the characteristic of k. To prove that K is not algebraically closed (and that it is far from being algebraically closed) we shall show that if $F(Z)$ is an element of $K[Z]$ of the form

$$F(Z) = Z^n - g(x) - h(y),$$

where $g(x)$ is an arbitrary element of $k[x]$ with $g(0)\neq 0$, $h(y)$ is an arbitrary element of $k[y]$ with $h(0)\neq 0$, and n is an integer greater than one with $n\not\equiv 0(p)$ if $p\neq 0$ and otherwise arbitrary, then $F(Z)$ has no linear factor in $K[Z]$ (and hence in particular if n is prime then $F(Z)$ is irreducible in $K[Z]$). *We shall prove the stronger assertion that $F(Z)$ has no linear factor in $K_1[Z]$*, and in proving this we shall let $g(x)$ be an arbitrary element of $k[[x]]$ with $g(0)\neq 0$ and $h(y)$ an arbitrary element of $k[[y]]$ with $h(0)\neq 0$ (i.e., $g(x)$ and $h(y)$ are power series respectively in x and y with coefficients in k and of positive leading degrees). Assume the contrary, i.e., that $F(Z)$ has a linear factor $Z-q$ in $K_1[Z]$. Then $q^n=g(x)+h(y)$. Since q is in the compositum K_1 of M_1^* and N_1^*, we can write $q=(\sum_{i=1}^m a_ib_i)/(\sum_{i=1}^m c_id_i)$ with $a_i, c_i\in M_1^*$ and $b_i, d_i\in N_1^*$ for $i=1, 2, \cdots, m$, where m is some positive integer. Therefore, there exist finite algebraic extensions M_2 and N_2 of M_1 and N_1 respectively such that $a_i, c_i\in M_2$ and $b_i, d_i\in N_2$ for $i=1, 2, \cdots, m$. Let r^* and s^* be the leading degrees of $g(x)$ and $h(y)$ respectively. Let $[M_2:M_1]=r$ and $[N_2:N_1]=s$. By extending M_2 and N_2 suitably, we can arrange matters so that $rr^*=ss^*=t$ say. Then $M_2=k((u))$ and $N_2=k((v))$ with $x=u^re(u)$ and $y=v^sf(v)$ where $e(u)$ and $f(v)$ are units in $k[[u]]$ and $k[[v]]$ respectively. Let H be the compositum of M_2 and N_2. Let $g^*(u)$ and $h^*(v)$ be the elements of $k[[u]]$ and $k[[v]]$ gotten by substituting $u^re(u)$ and $v^sf(v)$ for x and y in $g(x)$ and $h(y)$ respectively. Then the polynomial

$$F^*(Z) = [Z^n - d^*(u, v)] \in H[Z]$$

where $d^*(u, v)=g^*(u)+h^*(v)\in k[[u, v]]$, has a linear factor in $H[Z]$. *We shall prove that* this is a contradiction by proving more strongly that *$F^*(Z)$ has no linear factor in $L_2[Z]$ where $L_2=k((u, v))$*. Let $d(u, v)$ be the leading form of $d^*(u, v)$. Now, the leading degree of $g^*(u)$ in $u=rr^*=t=ss^*=$the leading degree of $h^*(v)$ in v. Therefore, $d(x, y)=au^t+bv^t$ where a and b are nonzero elements of k. Suppose, if possible, that $F^*(Z)$ has a linear factor $Z-w^*(u, v)$ with $w^*(u, v)\in L_2$.

Since $k[[u, v]]$ is integrally closed in L_2, we must have $w^*(u, v) \in k[[u, v]]$. Let $w(u, v)$ be the leading form of $w^*(u, v)$. Then $w^*(u, v)^n = d^*(u, v)$ and hence $w(u, v)^n = au^t + bv^t$; this is a contradiction (since $n \not\equiv 0(p)$ if $p \neq 0$).

2. **Finitely generated extensions.** Let now k be an algebraically closed field and let K be a finitely generated field extension of k. Let T be the compositum of all the algebraically closed subfields of K. Then we have that $T = k$ or equivalently we may state the

PROPOSITION. *Let L be an algebraically closed subfield of K. Then $L \subset k$.*

PROOF. Otherwise we can find an element x in L which is transcendental over k. Let P be the prime field of k and let P^* be the algebraic closure of P (in K). Then $P^* \subset k \cap L$. Let $Q = P^*(x)$ and $Q^* =$ the algebraic closure of Q (in K). Then $Q^* \subset L$. Let M be the compositum of k and Q^*. Then M is finitely generated over k and hence $[M:k(x)] = n < \infty$. Now we may either use elementary properties of polynomials or alternatively we may use valuations.

To invoke valuations, let v be the valuation of $k(x)/k$ for which $v(x) = 1$, where v is normalized so as to have the additive group of integers as its value group. Let w be an extension of v to M. Then for any nonzero element a of M with $w(a) > 0$ we must have: $w(a) \geqq 1/n$. Now $Q^* \subset M$ implies that for any integer m there exists y_m in M with $y_m^m = x$. Then $w(y_m) = (1/m)\ v(x) = 1/m$ and hence $0 < w(y_m) < 1/n$ whenever $m > n$; this is a contradiction. Therefore $L \subset k$.

Alternatively, to use an elementary argument, let m be an integer greater than n. Then there exists an element y_m in $Q^* \subset M$ such that $y_m^m = x$. Now, the polynomial $X^m - x$ is irreducible in $k[X, x]$ (X an indeterminate). Therefore $X^m - x$ is also irreducible in $k(x)[X]$. Hence $n = [M:k(x)] \geqq [k(x)(y_m):k(x)] = m$, which is a contradiction. Therefore $L \subset k$.

We observe that the method of §2 (of valuations) can be applied to prove the assertions of §1 and conversely the method of §1 (of power series) can be applied to prove the proposition of §2.

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