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PRINCETON UNIVERSITY AND
STANFORD UNIVERSITY

THE ZEROS OF CERTAIN SINE-LIKE INTEGRALS

A. E. LIVINGSTON AND LEE LORCH

We establish here the monotonic character of the zeros (modulo 1) of

$$\int_x^\infty \frac{f(t)}{t}\,dt, \qquad x > 0, \tag{1}$$

where $f(t)$ satisfies the conditions

(C1). $f(t) \geqq 0$ for $0 \leqq t < 1$;
(C2). $f(t) \not\equiv 0$ on any subinterval of $0 \leqq t < 1$;
(C3). $f(t+n) = (-1)^n f(t)$ for $n = 1, 2, 3, \cdots$;
(C4). $f(t)/t$ is Lebesgue integrable on $0 \leqq t \leqq 1$.

It is clear that these conditions imply that the integral (1) has precisely one zero, say z_n, in the interval $n < x < n+1$.

Let C be defined (uniquely) by the conditions

$$2\int_0^C f(t)dt = \int_0^1 f(t)dt, \qquad 0 < C < 1. \tag{2}$$

Now,

(A) $z_n - n \geqq C$ for $n = 0, 1, 2, \cdots$,
(B) $z_n - n \to C$ as $n \to \infty$,

as was shown in [2], even more generally, with the factor $1/t$ of $f(t)$ in (1) replaced by a function denoted there by $g(t)$ of which $1/t$ is a special case. When $f(t) = \sin \pi t$, the sequence $\{z_n - n\}_0^\infty$ is decreasing, as Harry Pollard has shown, and I. I. Hirschman has observed that Pollard's proof applies equally well to the zeros of

$$\int_x^\infty g(t) \sin \pi t dt \tag{3}$$

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where $g(t)$ is completely monotonic in $0<t<\infty$ [3, pp. 409–411]. (Here $g(t)$ has a meaning different from the one in [2].)

We prove here the following result:

Let $f(t)$ satisfy the conditions (C1)–(C4), *and denote by z_n the unique zero of* (1) *in the interval $n<x<n+1$, $n=0, 1, 2, \cdots$. Then $z_n - n \downarrow C$, where C is defined by* (2).

In replacing $\sin \pi t$ in $\mathrm{si}(\pi x)$ by a more general function the above result extends Theorem 3.2 of [3] in one direction, while Hirschman's observation concerning (3) generalizes that theorem in another fashion by replacing $1/t$ in $\mathrm{si}(\pi x)$ by an arbitrary completely monotonic function $g(t)$.

In view of (B), it is only the monotonicity of the sequence $\{z_n - n\}_0^\infty$ that need be established. The formula

$$\int_x^\infty \frac{f(t)}{t}\, dt = \int_x^{1+x} f(t) \sum_{i=0}^\infty \frac{(-1)^i}{t+i}\, dt \tag{4}$$

is obtained by writing (1) in the natural way as a sum of integrals over subintervals of $[x, \infty]$ of length one, making a linear change of variable $t=t'+i$ in each of these, and then interchanging the order of summation and integration. The function represented by the infinite series in (4) is denoted customarily [1, p. 20 (6)] by $G(t)/2$. With this notation, we have

$$\int_{z_n}^{1+z_n} f(t)G(t)\, dt = 0. \tag{5}$$

Suppose that $G_0(t)$ is a non-negative increasing function of t for $0<t<\infty$. By the second mean-value theorem, (A), and (5),

$$(-1)^n \int_{z_n}^{1+z_n} f(t)G(t)G_0(t)\, dt = (-1)^n G_0(1+z_n) \int_{\xi_n}^{1+z_n} f(t)G(t)\, dt \leqq 0,$$

where $z_n < \xi_n < z_n + 1$. Thus, if there is an α, $n+C \leqq \alpha < n+1$, for which

$$\int_\alpha^{\alpha+1} f(t)G(t)G_0(t)\, dt = 0,$$

then, necessarily, $\alpha \leqq z_n$. Since

$$0 = \int_{z_{n+1}-1}^{1+z_{n+1}} f(t)G(t)\, dt = -\int_{-l+z_{n+1}}^{z_{n+1}} f(t)G(t+1)\, dt$$

$$= -\int_{-l+z_{n+1}}^{z_{n+1}} f(t)G(t)\, \frac{G(t+1)}{G(t)}\, dt,$$

it follows by this argument that $z_{n+1}-(n+1)\leqq z_n-n$, provided only that $G(t+1)/G(t)$ is increasing for $0<t<\infty$. We show now that this is the case.

Recalling the definition of $G(t)$, it is easy to verify that $G(t)+G(t+1)=2/t$ [1, p. 20 (7)]. Thus, we may accomplish our aim by showing that $tG(t)$ is decreasing for $t>0$.

FIRST PROOF. (This was suggested in conversation with M. Riesz.) We have [1, p. 20(2)]

$$tG(t) = 2t\int_0^1 \frac{r^{t-1}}{1+r}\,dr.$$

Integrating by parts,

$$tG(t) = 1 + 2\int_0^1 \frac{r^t}{(1+r)^2}\,dr, \tag{6}$$

from which the desired result follows immediately.

REMARKS. On successive differentiation, (6) shows that $tG(t)$ is completely monotonic, $0<t<\infty$. This is true also of $t^\delta G(t)$ for any $\delta<1$, since $t^\delta G(t)$ can be written as the product of the two completely monotonic functions $1/t^{1-\delta}$, $\delta<1$, and $tG(t)$. [That the product of two completely monotonic functions is also completely monotonic follows at once from the successive differentiation of that product by Leibniz's rule.]

Moreover, the restriction $\delta\leqq 1$ cannot be removed if the function $t^\delta G(t)$ is to be completely monotonic, $0<t<\infty$, since $t^\delta G(t)$ increases rather than decreases (as required for complete monotonicity), at least for some positive interval of values of t, for any $\delta>1$.

To see this, let $\delta=1+\epsilon$, $\epsilon>0$. Then

$$\begin{aligned}[t^{1+\epsilon}G(t)]' &= t^\epsilon[(1+\epsilon)G(t)+tG'(t)]\\ &= 2t^\epsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{t+n}\left\{1+\epsilon-\frac{t}{t+n}\right\},\end{aligned}$$

where the infinite series representation is found directly or is taken from [1, p. 20 (6), p. 45 (10)].

This series is an alternating series whose first term is positive. The series itself will be shown to be positive for certain values of t, and the function $t^{1+\epsilon}G(t)$ to be increasing there, once we show that the terms of that series are monotonically decreasing for those values of t. Now, we observe that this is the case if

$$(7) \qquad t\left\{\frac{2t+2n+1}{(t+n+1)(t+n)}\right\} \leqq 1+\epsilon, \qquad n = 0, 1, 2, \cdots.$$

We note that the expression in braces decreases as n increases. Thus, the left member of (7) is greatest when $n=0$, i.e., its maximum is $(2t+1)/(t+1)$. But this maximum is $\leqq 1+\epsilon$ when $t^{-1} \geqq (1-\epsilon)/\epsilon$ and so we have shown that $t^{1+\epsilon}G(t)$ cannot be completely monotonic, $0<t<\infty$, for any $\epsilon>0$ whatever.

Some interest may attach to the above observation, since $G(t)$ is a standard "special function" and can be defined in terms of $\psi(t)$, the logarithmic derivative of the gamma function. Doing so, we can express these results as follows:

The function $t^{\delta}[\psi(t+1/2)-\psi(t)]$ *is completely monotonic,* $0<t<\infty$, *if and only if* $\delta \leqq 1$.

If $1<\delta<2$, *the function increases for* $0<t\leqq(\delta-1)/(2-\delta)$. *if* $\delta \geqq 2$, *it increases for all* $t>0$.

In case $\delta=1$, this shows [1, p. 20 (6)] that the hypergeometric function ${}_2F_1\ (1, t; 1+t; -1)$ is also completely monotonic, $0<t<\infty$.

SECOND PROOF. To show alternatively that $tG(t)$ decreases as t increases, $0<t<\infty$, we put $2t=1/s$ and use [1, p. 20 (6)], whence

$$2tG(2t) = 2\frac{1}{s}\sum_{n=0}^{\infty}\frac{(-1)^n}{1/s+n} = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{1+ns} = 2\int_0^1 \frac{dr}{1+r^s}.$$

The last expression clearly decreases from 2 to 1 as s decreases from ∞ to 0, that is, as t increases from 0 to ∞.

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UNIVERSITY OF WASHINGTON AND
PHILANDER SMITH COLLEGE