

## SOLUTION OF A PROBLEM BY ERDÖS- GILLMAN-HENRIKSEN

ABRAHAM ROBINSON

The following question is raised in [1, 5.1].

“Is a nondenumerable real-closed field—in particular if it is non-archimedean—characterized by its type of order as an ordered set?”

I shall show by means of a suitable example that the answer to this question is in the negative. More precisely, I shall specify two algebraically isomorphic real-closed non-archimedean and nondenumerable ordered fields, such that no order-preserving isomorphism exists between the two fields. Although it will be seen that the example is rather simple, there seems to be some point in putting this result on record, not only because of the distinguished parentage of the problem, but also because it underscores the significance of [1], in which the existence of order-preserving isomorphisms is ensured by more stringent conditions.

Let  $R$  be the field of rational numbers and let  $t_1$  and  $t_2$  be any pair of algebraically independent positive real transcendental numbers. Let  $R_1$  and  $R_2$  be the real closures (smallest real-closed extensions) of  $R(t_1)$  and  $R(t_2)$ , respectively. Then  $R_1$  and  $R_2$  are algebraically isomorphic under a continuation of the correspondence  $t_1 \leftrightarrow t_2$  and similarly ordered (of order type  $\eta$ ).

Consider the field,  $R_1^*$ , of fractional power series

$$(1) \quad a = \sum_{j=-m}^{\infty} a_j x^{j/n}, \quad a_j \in R_1, \quad n = 1, 2, 3, \dots, m = 0, 1, 2, \dots$$

Then  $R_1^*$  is real-closed, since  $R_1^*(i)$  is an algebraically closed proper extension of  $R_1^*$ .

Similarly, define  $R_2^*$  by (1) with  $a_j \in R_2$ , so that  $R_2$  is again a real-closed field. Both  $R_1^*$  and  $R_2^*$  are ordered by the rule that  $a > 0$  if  $a_j > 0$  for the nonvanishing  $a_j$  with lowest subscript; they are algebraically isomorphic under a correspondence for which  $t_1 \leftrightarrow t_2$  and  $x^k \leftrightarrow x^k$ ,  $k$  rational; and they are of the same order type since the latter depends only on the order types (and not on the arithmetic) of  $R_1$  and  $R_2$ , and these coincide. Finally, both  $R_1^*$  and  $R_2^*$  are non-archimedean and of cardinal  $c$ .

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Now suppose that there exists an order-preserving isomorphism between  $R_1^*$  and  $R_2^*$ . Let  $t_1^*$  be the image of  $t_1$  in  $R_2^*$ . Then we must have, in  $R_2^*$ ,

$$a > t_1^* \text{ for all rational numbers } a > t_1,$$

and

$$a < t_1^* \text{ for all rational numbers } a < t_1.$$

This implies that in the representation of  $t_1^*$  by (1) we have

$$(2) \quad t_1^* = \sum_{j=0}^{\infty} a_j x^{j/n} \quad a_j \in R_2 \text{ and } a_0 > 0.$$

Now  $t_1$  is not contained in  $R_2$ , and the ordering of  $R_2$  is archimedean. Accordingly, *one* of the two following alternatives must apply.

(i) There exists a rational number  $a$  such that

$$a_0 < a < t_1.$$

or (ii) there exists a rational number  $a$  such that

$$a_0 > a > t_1.$$

In either case, consider the image of  $t_1 - a$  in  $R_2^*$ . This is

$$t_1^* - a = (a_0 - a) + \sum_{j=1}^{\infty} a_j x^{j/n}.$$

Then, in case (i),  $t_1 - a > 0$  in  $R_1^*$ , while the image of  $t_1 - a$  in  $R_2^*$  is negative. On the other hand, in case (ii),  $t_1 - a < 0$  in  $R_1^*$ , while the image of  $t_1 - a$  in  $R_2^*$  is positive. We conclude that no order-preserving isomorphism exists between  $R_1^*$  and  $R_2^*$ .

REFERENCE

1. P. Erdős, L. Gillman and M. Henriksen, *An isomorphism theorem for real-closed fields*, Ann. of Math. vol. 61 (1955) pp. 542-554.

UNIVERSITY OF TORONTO