5. L. K. Hua, *A note on the total matrix ring over a non-commutative field*, Annales de la Société Polonaise de Mathématique vol. 25 (1952) pp. 188–198.

6. N. Jacobson and C. E. Rickart, *Jordan homomorphisms of rings*, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 479–502.

7. F. Kasch, *Invariante Untermoduln des Endomorphismenrings eines Vektorraums*, Archiv der Mathematik vol. 4 (1953) pp. 182–190.

8. ———, *Über den Automorphismenring einfacher Algebren*, Archiv der Mathematik vol. 6 (1955) pp. 59–65.

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CANCELLATION IN DIRECT SUMS OF GROUPS

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1. Introduction. The purpose of this paper is to consider the following question for groups. If $F \oplus G = F' \oplus H$ and $F \cong F'$, when is $G \cong H$? It is easy to see that for G to be isomorphic to H some additional hypothesis must be given. For example let $C_i, i = 1, 2, 3, \dots$ be cyclic of order two. Let $F = C_2 \oplus C_3 \oplus C_4 \oplus \dots$, let $G = C_1$, let $F' = C_1 \oplus C_3 \oplus C_5 \oplus \dots$, and let $H = C_2 \oplus C_4 \oplus C_6 \oplus \dots$. Clearly $F \oplus G = F' \oplus H$ and $F \cong F'$, yet $G \not\cong H$. One hypothesis that is suggested by this example is to require that F be finitely generated. In fact, in Kaplansky's book, *Infinite Abelian groups*, p. 13, the author asks the following question for Abelian groups, called Test Problem III. If $F \oplus G = F' \oplus H$, $F \cong F'$, and F is finitely generated, is $G \cong H$? The main results of this paper are theorems from which Test Problem III follows as a corollary.

2. Notations. The additive notation for groups will be used. The symbol $F \oplus G$ will denote the direct sum of the two groups F and G . The commutator subgroup of a group G will be denoted by $Q(G)$. The symbol $\{S\}$ will denote the group generated by the set S of elements. The order of an element g of G will be written $o(g)$, and $o(S)$ will denote the number of elements in the set S . The infinite cyclic group will be represented by C , and the cyclic group of order n by $C(n)$. The symbol $Z(G)$ will denote the center of the group G .

3. Definitions. Suppose $F \oplus G = F' \oplus H$. The set of F components of the elements of F' will be denoted by F_1 , and the set of F com-

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ponents of the elements of H will be denoted by F_2 . The sets G_1 , G_2 , F'_1 , F'_2 , H_1 , and H_2 are defined analogously. All these sets are normal subgroups of $F \oplus G$.

4. Theorem. *If $F \oplus G = F' \oplus H$ then $F + H = F \oplus G_2 = F'_1 \oplus H$.*

PROOF. It is easy to see that $F + H = F \oplus ((F + H) \cap G)$. Let g be in $(F + H) \cap G$. Then $g = f + h$ with f in F and h in H . But $h = f_2 + g_2$ with f_2 in F_2 and g_2 in G_2 . Hence $g = f + f_2 + g_2$. Since any element can be written in exactly one way as the sum of an element of F and one of G , it follows that $f + f_2 = 0$ and $g = g_2$. Thus $(F + H) \cap G \subseteq G_2$. Let g_2 be any element of G_2 . Then $(f + g_2)$ is in H for some f in F . Then $-f + (f + g_2) = g_2$ is in G_2 so that $G_2 \subseteq F + H$. Hence $G_2 \subseteq (F + H) \cap G$, and we get $F + H = F \oplus G_2$. Similarly $F + H = F'_1 \oplus H$.

5. Theorem. *If $F \oplus G = F' \oplus H$, then $F/(F \cap H) \cong F'_1$, and $H/(F \cap H) \cong G_2$.*

PROOF. Project F onto its F' components. The range of this projection is F'_1 and the kernel is $F \cap H$. Hence $F/(F \cap H) \cong F'_1$. Similarly $H/(F \cap H) \cong G_2$.

6. Theorem. *If $F \oplus G = F' \oplus H$, $F \cong F'$, and F is finite then $G \cong H$.*

PROOF. By §4, $F + H = F \oplus G_2 = F'_1 \oplus H$. If $F \cap H = 0$, then $F \oplus H$ exists and $F \oplus H = F'_1 \oplus H$, so that $F \cong F'_1$. Since F is finite and $F \cong F'$, it follows that $F'_1 = F'$ and $F \oplus H = F' \oplus H = F \oplus G$. Hence $G \cong H$. If $F \cap H \neq 0$, we proceed by induction on $o(F) = o(F')$. If $o(F) = 1$ the theorem is obvious. Since $F \oplus G = F' \oplus H$ we get that $F/(F \cap H) \oplus G \cong F' \oplus H/(F \cap H)$. $F' \cong F$ and $H/(F \cap H) \cong G_2$ by §5 implies that $F/(F \cap H) \oplus G \cong F \oplus G_2 = F'_1 \oplus H$. By 5, $F/(F \cap H) \cong F'_1$ and since $F \cap H \neq 0$, it follows that $o(F'_1) < o(F)$. Hence our induction hypothesis implies that $G \cong H$.²

7. Theorem. *If $F \oplus G = F' \oplus H$, $F \cong F' \cong C$, and G is Abelian, then $G \cong H$.*

PROOF. Let P be the projection $(f' + h)P = f'$, where f' is in F' and h is in H . Then P is a homomorphism of G into F' with kernel $G \cap H$. If $GP = 0$, then $G \subseteq H$ and since G is a direct summand of a group containing H , we get $H = G \oplus H'$ for some group H' . Hence $F \oplus G = F' \oplus G \oplus H'$ and $F \cong F' \oplus H'$. Since $F \cong F' \cong C$, it follows that $H' = 0$ and $G = H$. If $GP \neq 0$ then GP is infinite cyclic, so that $G/(G \cap H) \cong C$, from which follows that $G \cong (G \cap H) \oplus C$ since G is

² Bjarni Jonsson has informed me that this result follows from Theorem 3.11 in the book *Direct Decomposition of Finite Algebraic Systems*, by Jonsson and Tarski.

Abelian. The group H is also Abelian so by symmetry we may assume $H \cong (H \cap G) \oplus C$. Hence $G \cong H$.

8. Corollary (Kaplansky's Test Problem III). If $F \oplus G = F' \oplus H$, $F \cong F'$ and F is finitely generated, and F and G are Abelian, then $G \cong H$.

PROOF. Since F is Abelian and finitely generated it is the direct sum of a finite group and a finite number of infinite cyclic groups. Clearly it suffices to prove $G \cong H$ for the two cases F finite and F infinite cyclic. But these two cases are proved in §§6 and 7. Hence §8 follows.

9. Corollary. The decomposition of a finitely generated Abelian group into indecomposable summands is unique up to isomorphism.

PROOF. Let $F_1 \oplus F_2 \oplus \cdots \oplus F_m = F'_1 \oplus F'_2 \oplus \cdots \oplus F'_n$ be two decompositions of a finitely generated Abelian group into indecomposable summands. Then each F_i and F'_k is cyclic of prime power order or infinite cyclic. We induct on m . If $m = 1$, then §9 follows because F_i and F'_k are indecomposable. Suppose the left side has an infinite cyclic summand. Then so does the right. Let for convenience these two summands be F_1 and F'_1 . Then by §7, $F_2 \oplus F_3 \oplus \cdots \oplus F_m \cong F'_2 \oplus F'_3 \oplus \cdots \oplus F'_n$ and by the induction hypothesis §9 follows.

Now we may assume all the summands on the left side are finite. Let F_i be a summand of maximal prime power order, say p^r . Hence F_i has an element f_i of order p^r . Furthermore $f_i = f'_1 + f'_2 + \cdots + f'_n$ and some f'_j is of order p^r . If $o(f'_j) < o(F'_j)$ then the right side has an element of order p^s with $r < s$ and the left does not. Hence $o(F'_j) = o(F_i)$ and by §6 and the induction hypothesis we are done.

10. Theorem. If $F \oplus G = F' \oplus H$, then the following are true:

- (a) $Q(F_1) \subseteq F \cap F'$.
- (b) $Q(F_2) \subseteq F \cap H$.
- (c) $Q(F) = Q(F_1) \oplus Q(F_2)$.
- (d) $Q(F) = Q(F'_1) \oplus Q(H_1)$.

PROOF. The proof is straightforward and routine, so is omitted.

11. Theorem. If $F \oplus G = F' \oplus H$, $F \cong F' \cong C$, then the following are true:

- (a) $Q(G) = Q(H)$.
- (b) $G/Q(G) \cong H/Q(H)$.
- (c) The group G is isomorphic to a normal subgroup of H of finite index in H . In fact $G \cong H$ or $G \cong H_2$.
- (d) The group H is isomorphic to a normal subgroup of G of finite index in G . In fact $H \cong G$ or $H \cong G_2$.

(e) If $Z(G)$ is torsion then $G \cong H$. (In particular, if G is finite or if $Z(G) = 0$, then $G \cong H$.)

(f) If $Q(G) = G$ then $G \cong H$.

PROOF. (a) Using §10 applied to G and H and the fact that F and F' are Abelian, we get that $Q(G) = Q(G_2)$ and $Q(H) = Q(H_2)$. But $Q(G) = Q(F'_2) \oplus Q(H_2) = Q(H_2)$ implies $Q(G) = Q(H)$.

(b) Since $Q(G) = Q(H)$ we get $F \oplus G/Q(G) \cong F' \oplus H/Q(H)$. The groups $G/Q(G)$ and $H/Q(H)$ are Abelian. By §7, $G/Q(G) \cong H/Q(H)$.

(c)—(d) Let $F = \{f\}$ and $F' = \{nf + g\}$. Then $f = (mnf + mg) + ((1 - mn)f - mg)$, with $mnf + mg$ in F' and $((1 - mn)f - mg)$ in H . If $n = 0$ then $F' \subseteq G$. In that case $G = F' \oplus G'$, for some G' . Thus $F \oplus F' \oplus G' = F' \oplus H$ and so $H \cong F \oplus G' \cong F' \oplus G'$. Hence $G \cong H$. If $m = 0$ then $F \subseteq H$, and similarly we get $G \cong H$. If $mn = 1$ then $F \cap H = 0$ and the projection of H onto G_2 is an isomorphism. Since $\{G_2, g\} = G$, and mg is in G_2 , it follows that the index of G_2 in G is finite. If $mn \neq 1$ and $mn \neq 0$, then $F \cap H = 0$ because $o(f)$ is infinite and so the F' component of no nonzero element of F is 0. As in the previous case, $H \cong G_2$. By symmetry we may likewise show that $G \cong H$ or $G \cong H_2$. Thus (c) and (d) are proved.

(e) In proving (c) and (d) it was shown that either $G \cong H$ or $F \cap H = 0$. If $F \cap H = 0$ then since $((1 - mn)f - mg)$ is the H component of f , it follows that $o(g)$ is infinite. But $F' = \{nf + g\}$ implies that g is in $Z(G)$. Thus if $Z(G)$ is torsion, $F \cap H \neq 0$ and $G \cong H$.

(f) If $Q(G) = G$, then by (b), $Q(H) = H$. Using (a), we get $G = H$.

12. **Remark.** Professor W. R. Scott constructed the interesting example given in the next theorem.

13. **Theorem.** *There exist groups G and H such that G not $\cong H$, yet $F \oplus G = F' \oplus H$ with $F \cong F' \cong C$.*

PROOF. Let $G \cap H = \{x\}$ with $11x = 0$. Let G be the extension of $G \cap H$ by the automorphism A such that $xA = 2x$. That is, $G = \{x, y\}$ where $-y + x + y = 2x$ and $G/(G \cap H) \cong \{y\} \cong C$. Let H be the extension of $G \cap H$ by the automorphism A such that $xA = 8x$. That is, $H = \{x, z\}$ where $-z + x + z = 8x$ and $H/(G \cap H) \cong \{z\} \cong C$. Now let $F = \{7y + z\}$, $F' = \{y + 3z\}$, and $z + y = y + z$. It is straightforward to prove that the groups just defined have the desired properties.

14. **Theorem.** *If $F \oplus G = F' \oplus H$, $F \cong F'$, F is indecomposable, and $Q(F) = F$, then $G \cong H$.*

PROOF. By §10(c), $Q(F) = Q(F_1) \oplus Q(F_2)$. Since F is indecomposable, $Q(F) = Q(F_1)$ and $Q(F_2) = 0$, or $Q(F) = Q(F_2)$ and $Q(F_1) = 0$. In the first

case $F \subseteq F'$ and by §10(d), $F = Q(F) = Q(F'_1)$. Since $F \cong F'$ is indecomposable and $F' = Q(F') = Q(F'_1) \oplus Q(F'_2)$ it follows that $F = F'$ and that $G \cong H$. Similarly if $Q(F) = Q(F_2)$ and $Q(F_1) = 0$ then $G \cong H$.

15. **Corollary.** *If $F \oplus G = F' \oplus H$, $F \cong F'$ and F is simple, then $G \cong H$.*

PROOF. If F is non-Abelian, §15 follows from §14. If F is Abelian §15 follows from §6.

16. **Concluding remarks.** The theorems given in §§4, 5, and 6 can be generalized to operator groups. In §6, instead of requiring in the operator group case that F be finite, one requires that F have only a finite number of admissible subgroups. Kaplansky's Test Problem III may be extended to modules over principal ideal rings, and if F is torsion, to modules over Dedekind rings. The proof given in §9 of the uniqueness of the decomposition of finitely generated Abelian groups is, in this writer's opinion, a much better proof than has existed heretofore.

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