A SEPARABLE NORMAL NONPARACOMPACT SPACE

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A topological space X is said to be *paracompact* [1] if for every open covering G of X there is a locally finite open covering G' of X which is a refinement of G. (G' is *locally finite* if every point of X has a neighborhood which intersects only a finite number of members of G'.) It is known that every paracompact Hausdorff space is normal [1] and that every metrizable space is paracompact [2]. Since every normal Hausdorff space with a countable base is metrizable, therefore, *every normal Hausdorff space with a countable base is paracompact.*

The purpose of this paper is to show that the existence of a countable base cannot be replaced by separability in this last statement.

THEOREM. *There exists a separable normal Hausdorff space X which is not paracompact and does not have the Lindelöf property.*

In this paper Greek letters will always denote countable ordinals; and the letters i, j, k, m, and n will stand for positive integers.

CONSTRUCTION OF X. Let A denote the set of all ordered pairs (m, n) and B the set of all countable ordinals. Then P will be a point of X if and only if P is a term of A or B.

For each α, let f_α be a function defined on the positive integers, with values in the positive integers, such that: if $\alpha<\beta$, there exists an integer $m(\alpha,\beta)=m(\beta,\alpha)$ such that $f_\alpha(i)<f_\beta(i)$ whenever $i>m(\alpha,\beta)$. These functions are used to define the topology of X.

The set N will be a neighborhood if and only if it belongs to one of the following classes.

(1) Every point of A is a neighborhood of itself.

(2) If α is not a limit ordinal, then corresponding to each n there is a neighborhood of α which consists of (a) α itself, and (b) all pairs $(k, f_\alpha(k))$ with $k>n$.

(3) If α is a limit ordinal, choose a $\beta<\alpha$, and an $n(\gamma)$ for each γ such that $\beta<\gamma\leqq\alpha$. For each such collection of choices there is a neighborhood of α which consists of:

(a) all γ such that $\beta<\gamma\leqq\alpha$, and

(b) all pairs $(k, f_\gamma(k))$ with $k>n(\gamma)$ and $\beta<\gamma\leqq\alpha$.

This completes the construction of X.

Since A is dense in X, X is separable; and since any two distinct points have disjoint neighborhoods, X is a Hausdorff space.

Presented to the Society, August 13, 1955; received by the editors May 19, 1955.

PROOF THAT X IS NORMAL. Let H and K be closed and disjoint subsets of X. If both H and K are uncountable, then $H \cdot B$ and $K \cdot B$ are both uncountable, and there exist sequences $\beta_1, \beta_2, \beta_3, \cdots$ and $\alpha_1, \alpha_2, \alpha_3, \cdots$ such that for each n, β_n belongs to H, α_n belongs to K, and $\beta_n < \alpha_n < \beta_{n+1}$. The common limit ordinal is a limit point of both H and K; but H and K have no common limit point.

Hence at least one of these sets, say H, is countable. Choose α_0 such that, if α is in H, $\alpha < \alpha_0$. The construction of disjoint open sets covering H and K will be carried out with the aid of the integers $n(\alpha)$ now to be defined for each α.

I. For $\alpha > \alpha_0$, choose $n(\alpha) > m(\alpha_0, \alpha)$.

II. Order the ordinals which do not exceed α_0 in a simple countable sequence $\alpha_0, \alpha_1, \alpha_2, \cdots$. Take $n(\alpha_0) = 1$. Then having chosen $n(\alpha_0), \cdots, n(\alpha_{i-1})$, choose $n(\alpha_i) > m(\alpha_i, \alpha_j)$ where $0 \leqq j < i$.

Suppose that α belongs to K. There exists an ordinal β which is maximal with respect to two properties: (a) β is in H, and (b) $\beta < \alpha$. (It is assumed here that H contains ordinals less than α; in the contrary case take $\beta = 1$.) If α is a limit ordinal construct the neighborhood $U'(\alpha)$ in accordance with (3) using the β above and the $n(\gamma)$'s described in I and II. If α is not a limit ordinal construct the neighborhood $U'(\alpha)$ in accordance with (2), taking $n = n(\alpha)$. In either case, let $U(\alpha)$ be a neighborhood of α which does not intersect $H \cdot A$ such that $U(\alpha)$ is a subset of $U'(\alpha)$.

If α is in H, carry out the same construction interchanging H and K to obtain a neighborhood $V(\alpha)$. Let U and V be the sum of all neighborhoods $U(\alpha)$ and $V(\alpha)$ for α in K and H, respectively.

Let $U' = U + (K \cdot A)$ and $V' = V + (H \cdot A)$. Then U' and V' are open and disjoint coverings of K and H, respectively.

PROOF THAT X IS NOT PARACOMPACT. Let G be the set of all neighborhoods defined in (1), (2), and (3). Note that every member of G is countable. If G' is any refinement of G covering X, some term of A is in uncountably many members of G'. Hence X is not paracompact.

The same covering G shows that X does not have the Lindelöf property, since no countable subcollection of G can cover X.

REFERENCES

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