# ON CLOSURE PROBLEMS AND THE ZEROS OF THE RIEMANN ZETA FUNCTION[1]

NORMAN LEVINSON

1. In the memoirs of Wiener [7] on Tauberian theorems it is pointed out that the closure of the translations in $L(-\infty, \infty)$ of

$$e^{(\sigma-1)x}\frac{d}{dx}\left(\frac{e^x}{e^{e^x}-1}\right)$$

is a necessary and sufficient condition for the Riemann zeta function $\zeta(s)$ to have no zeros on the line $\operatorname{Re} s=\sigma$, $0<\sigma<1$.

Salem [4] using

$$(1-2^{1-s})\zeta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$$

in place of $\zeta(s)$ shows that another necessary and sufficient condition is that, if $f(x)$ is a bounded measurable function on $(0, \infty)$, then

$$\int_0^\infty \frac{x^{\sigma-1}}{e^{ax}+1}f(x)dx = 0$$

for all $a$ $(0<a<\infty)$ should imply that $f$ is zero almost everywhere.

Here somewhat different conditions will be considered.

THEOREM I. *Let $\lambda_n$ be a positive increasing sequence such that*

$$(1.0)\qquad \sum^{\infty}\frac{1}{\lambda_n}=\infty.$$

*A necessary and sufficient condition that $\zeta(s)$ have no zeros in the strip $\sigma_1<\operatorname{Re} s<\sigma_2$, where $1/2\leqq\sigma_1<\sigma_2\leqq 1$, is that given any $\epsilon>0$ and $\alpha$ and $\beta$ such that $\sigma_1<\alpha<\beta<\sigma_2$ there exists an integer $N$ and $\{a_n\}$, $n=1, \cdots, N$, (depending on $\epsilon$, $\alpha$ and $\beta$) such that*

$$(1.1)\qquad \int_0^\infty\left(\sum_1^N a_n\frac{e^{-\lambda_n x}}{1+e^{-\lambda_n x}}-e^{-x}\right)^2(x^{2\alpha-1}+x^{2\beta-1})dx<\epsilon.$$

A particular case of the above is with $\lambda_n=n$.

REMARK. It is rather trivial to show that if $(x^{2\alpha-1}+x^{2\beta-1})$ in (1.1)

Received by the editors September 1, 1955 and, in revised form, October 24, 1955.

[1] Sponsored in part by the Office of Naval Research.

is replaced by $x^{2c-1}$ for any $c$, $1/2 \leqq c \leqq 1$, then the left side of (1.1) can always be made less than $\epsilon$ regardless of the location of zeros of $\zeta(s)$. (See end of paper.)

A result equivalent to Theorem I is the following.

THEOREM II. *A necessary and sufficient condition that $\zeta(s)$ have no zeros in the strip $\sigma_1 < \mathrm{Rl}\ s < \sigma_2$ is that for any $f(x) \in L^2(0, \infty)$ and $\alpha$ and $\beta$ such that $\sigma_1 < \alpha < \beta < \sigma_2$,*

$$(1.2) \qquad \int_0^\infty \frac{e^{-\lambda_n x}}{1 + e^{-\lambda_n x}} (x^{\alpha - 1/2} + x^{\beta - 1/2}) f(x) dx = 0, \qquad n = 1, 2, \cdots$$

*implies that $f(x)$ is zero almost everywhere on $(0, \infty)$. Here $\lambda_n$ satisfies* (1.0) *and* $1/2 \leqq \sigma_1 < \sigma_2 \leqq 1$.

An immediate consequence of Theorem I is that a sufficient condition for $\zeta(s)$ to have no zeros in the strip $(\sigma_1, \sigma_2)$ is that (1.1) hold with $\alpha = \sigma_1$ and $\beta = \sigma_2$. Similarly an immediate consequence of Theorem II is that a sufficient condition for $\zeta(s)$ to have no zeros in the strip $(\sigma_1, \sigma_2)$ is that (1.2), with $\alpha = \sigma_1$ and $\beta = \sigma_2$, should imply $f(x)$ zero almost everywhere. In the case of Theorem II this follows from the fact that

$$\frac{x^{\alpha - 1/2} + x^{\beta - 1/2}}{x^{\sigma_1 - 1/2} + x^{\sigma_2 - 1/2}}$$

is bounded on $(0, \infty)$ and of Theorem I from the boundedness of

$$(x^{2\alpha - 1} + x^{2\beta - 1})/(x^{2\sigma_1 - 1} + x^{2\sigma_2 - 1}).$$

It has been pointed out to the author that these results can be derived with the aid of [**1**; **2**; **3**]. However it appears desirable to give a self-contained derivation.

2. The proof that (1.1) is a sufficient condition for $\zeta(s)$ to have no zeros in the strip $(\sigma_1, \sigma_2)$ is simple. Indeed for Re $s > 0$

$$(2.0) \qquad \zeta(s)(1 - 2^{1-s})\Gamma(s) = \int_0^\infty \frac{e^{-x}}{1 + e^{-x}} x^{s-1} dx.$$

Let $\zeta(s_0) = \zeta(\sigma_0 + it_0) = 0$ where $\sigma_1 < \sigma_0 < \sigma_2$. Then by (2.0) setting $x = \lambda_n y$ there follows

$$(2.1) \qquad \int_0^\infty \frac{e^{-\lambda_n y}}{1 + e^{-\lambda_n y}} y^{\sigma_0 + it_0 - 1} dy = 0.$$

Take $c$ small enough so that $\sigma_1 < \sigma_0 - c < \sigma_0 + c < \sigma_2$ and take $\alpha = \sigma_0 - c$ and $\beta = \sigma_0 + c$. Then from (1.1)

$$\int_0^\infty \left( \sum_1^N a_n \frac{e^{-\lambda_n y}}{1 + e^{-\lambda_n y}} - e^y \right)^2 (y^{2\sigma_0 - 2c - 1} + y^{2\sigma_0 + 2c - 1}) dy < \epsilon$$

which gives

$$(2.2) \qquad \int_0^1 \left( \sum_1^N a_n \frac{e^{-\lambda_n y}}{1 + e^{-\lambda_n y}} - e^y \right)^2 y^{2\sigma_0 - 2c - 1} dy < \epsilon,$$

$$(2.3) \qquad \int_1^\infty \left( \sum_1^N a_n \frac{e^{-\lambda_n y}}{1 + e^{-\lambda_n y}} - e^y \right)^2 y^{2\sigma_0 + 2c - 1} dy < \epsilon.$$

From (2.1)

$$(2.4) \quad -\Gamma(\sigma_0 + it_0) = \int_0^\infty \left( \sum_1^N a_n \frac{e^{-\lambda_n y}}{1 + e^{-\lambda_n y}} - e^{-y} \right) y^{\sigma_0 + it_0 - 1} dy.$$

Writing the integral above as an integral over (0, 1) plus one over (1, $\infty$) and using the Schwartz inequality it follows from (2.2) and (2.3) that

$$\left| \Gamma(\sigma_0 + it_0) \right| \leqq \epsilon^{1/2} \left( \int_0^1 y^{2c-1} dy \right)^{1/2} + \epsilon^{1/2} \left( \int_1^\infty y^{-2c-1} dy \right)^{1/2}$$
$$= \left( \frac{2\epsilon}{c} \right)^{1/2}.$$

Since $\epsilon$ can be taken arbitrarily small and $\Gamma(\sigma_0 + it_0) \neq 0$ this is impossible. Thus $\zeta(s)$ cannot vanish[2] in the strip $\sigma_1 < \mathrm{Rl}\, s < \sigma_2$.

3. Here the necessity of the condition of Theorem II will be proved; that is, it will be shown that if $\zeta(s)$ has no zeros in $(\sigma_1, \sigma_2)$ then (1.2) implies $f(x)$ is zero.

First it will be shown that (1.2) implies that if

$$(3.0) \qquad H(w) = \int_0^\infty \frac{e^{-wx}}{1 + e^{-wx}} (x^{\alpha - 1/2} + x^{\beta - 1/2}) f(x) dx$$

then for Re $w > 0$,

$$(3.1) \qquad H(w) = 0.$$

Let $w = u + iv$. Let $c > 0$. For $u \geqq c$ and $0 < x < 1/|v|$

$$\mathrm{Re}\,(1 + e^{-wx}) = 1 + e^{-ux} \cos vx \geqq 1 + e^{-ux} \cos 1 \geqq 1.$$

[2] The trivial character of all such sufficiency proofs seems to indicate that if the Riemann hypothesis is true the closure theorems do not seem to be a very promising direction to pursue.

For $x \geqq 1/|v|$

$$\operatorname{Re}(1 + e^{-wx}) \geqq 1 - e^{-ux} \geqq 1 - e^{-u/|v|} \geqq 1 - e^{-c/|v|}.$$

Thus for all $x > 0$, $u \geqq c$,

$$|1 + e^{-wx}| \geqq 1 - e^{-c/|v|}. \tag{3.2}$$

For $|v| \leqq c$, $1 - e^{-c/|v|} \geqq 1 - e^{-1} > 1/2$ and for $|v| \geqq c$, $1 - e^{-c/|v|} \geqq c/2|v|$. Thus for small $c$ it follows from (3.2) that

$$\frac{1}{|1 + e^{-wx}|} \leqq 2\,\frac{1 + |v|}{c}.$$

Therefore the integrand for $H(w)$ satisfies

$$\begin{aligned} &\left|\frac{e^{-wx}}{1 + e^{-wx}}\,(x^{\alpha-1/2} + x^{\beta-1/2})f(x)\right| \\ &\qquad\qquad \leqq \frac{4}{c}\,(1 + |v|)e^{-cx}|f(x)|\max(1, x^{\beta-1/2}). \end{aligned} \tag{3.3}$$

Using (3.3) in (3.0) and applying the Schwartz inequality it follows that the integral for $H(w)$ is uniformly convergent for $w$ in any bounded domain in $u \geqq c$. Thus $H(w)$ is analytic for $u > c$ and since $c$ is arbitrary it follows that $H(w)$ is analytic for $u > 0$. Also by the Schwartz inequality and (3.3)

$$\begin{aligned} |H(w)| \leqq \frac{4}{c}\,(1 + |w|)&\left(\int_0^\infty e^{-2cx}(1 + x^{2\beta-1})dx\right)^{1/2} \\ &\cdot\left(\int_0^\infty |f(x)|^2 dx\right)^{1/2}. \end{aligned}$$

In particular if $c = 1$

$$|H(w)| \leqq K|w|, \qquad u \geqq 1, \tag{3.4}$$

where $K$ is a constant. Applying an inequality of Carleman [6, p. 130] to $H(w)$ in the half-plane $u \geqq 1$ it follows that the sum of the reciprocals of the real zeros of $H(w)$ for $u > 2$ must converge unless $H$ is zero. But by (1.0) this proves (3.1).

LEMMA. *For any fixed real $p$ there exists a function $R(u)$ continuous for $u > 0$ and such that*

$$\int_0^\infty u^{-k}|R(u)|\,du < \infty \tag{3.5}$$

*for all k,* $\sigma_1 < k < \sigma_2$, *and*

$$\int_0^\infty \frac{e^{-ux}}{1+e^{-ux}} R(u)du = \exp\left(-\frac{1}{2}\log^2 x + ip\log x\right). \tag{3.6}$$

The proof of this lemma will be given in §4. Let

$$\begin{aligned} I &= \int_0^\infty R(u)H(u)du \\ &= \int_0^\infty R(u)du \int_0^\infty \frac{e^{-ux}}{1+e^{-ux}} (x^{\alpha-1/2} + x^{\beta-1/2})f(x)dx. \end{aligned} \tag{3.7}$$

Using the Schwartz inequality

$$\begin{aligned} J &= \int_0^\infty |R(u)| du \int_0^\infty \frac{e^{-ux}}{1+e^{-ux}} x^{\alpha-1/2} |f(x)| dx \\ &\leqq \int_0^\infty |R(u)| du \left(\int_0^\infty \left(\frac{e^{-ux}}{1+e^{-ux}}\right)^2 x^{2\alpha-1}dx\right)^{1/2} \left(\int_0^\infty |f(x)|^2 dx\right)^{1/2}. \end{aligned}$$

Since

$$\int_0^\infty \left(\frac{e^{-ux}}{1+e^{-ux}}\right)^2 x^{2\alpha-1}dx = u^{-2\alpha}\int_0^\infty \left(\frac{e^{-y}}{1+e^{-y}}\right)^2 y^{2\alpha-1}dy,$$

$$J \leqq C_1 \int_0^\infty u^{-\alpha} |R(u)| du$$

where $C_1$ is a constant. By (3.5) with $k=\alpha$ it follows that $J$ is bounded. The same proof holds with $\alpha$ replaced by $\beta$. Thus the repeated integral representing $I$ is absolutely convergent and the order of integration can be inverted. Doing this and using (3.6)

$$I = \int_0^\infty (x^{\alpha-1/2} + x^{\beta-1/2})f(x) \exp\left(-\frac{1}{2}\log^2 x + ip\log x\right)dx.$$

Setting $x = e^y$

$$I = \int_{-\infty}^\infty G(y)e^{ipy}dy \tag{3.8}$$

where

$$G(y) = (e^{\alpha y} + e^{\beta y})f(e^y)e^{y/2}e^{-y^2/2}.$$

Since $f(e^y)e^{y/2} \in L^2(-\infty, \infty)$ it follows from the Schwartz inequality that $G(y)$ is absolutely integrable. On the other hand since $H(u)=0$

it follows from (3.7) that $I=0$. Since this holds for all real $p$ and since, by (3.8), $I=I(p)$ is the Fourier transform of $G(y)$ it follows that $G(y)$ is zero almost everywhere and thus $f(x)$ must be zero almost everywhere, which proves the necessity of the condition of Theorem II for $\zeta(s)$ to be free of zeros in $(\sigma_1, \sigma_2)$.

4. Here the lemma will be proved. Let

$$R(u) = \frac{1}{i(2\pi)^{1/2}} \int_{-i\infty+c}^{i\infty+c} \frac{\exp((s+ip)^2/2)u^{s-1}}{\Gamma(s)\zeta(s)(1-2^{1-s})} ds \tag{4.0}$$

where $c$ is a constant, $\sigma_1 < c < \sigma_2$. It will be shown that $R(u)$ does not depend on $c$. Indeed let $\delta > 0$ and let $\sigma_1 + \delta \leqq c \leqq \sigma_2 - \delta$. It follows easily from familiar properties of $\zeta(s)$ [5, Theorem 9.6B] that if $\zeta(s)$ has no zeros in the strip $\sigma_1 < \text{Rl } s < \sigma_2$ then there is a constant $A$, which depends on $\delta$, such that if $s = \sigma + it$ then

$$|\zeta(s)| > (2 + |t|)^{-A}, \quad \sigma_1 + \delta \leqq \sigma \leqq \sigma_2 - \delta. \tag{4.1}$$

Also since $1/2 \leqq \sigma_1 < \sigma_2 \leqq 1$ it follows that

$$\left| \frac{\exp((s+ip)^2/2)}{\Gamma(s)(1-2^{1-s})} \right| < Ke^{-|t|}, \qquad \sigma_1 + \delta \leqq \sigma \leqq \sigma_2 - \delta \tag{4.2}$$

for some $K$ which depends on $\delta$ and $p$. Thus from (4.0)

$$|R(u)| \leqq \int_{-\infty}^{\infty} Ke^{-|t|}(2 + |t|)^A u^{c-1} dt.$$

Or, there is a $B$ depending on $\delta$ and $p$ such that

$$|R(u)| \leqq Bu^{c-1}. \tag{4.3}$$

That $R(u)$ does not depend on $c$ for $\sigma_1 + \delta \leqq c \leqq \sigma_2 - \delta$ follows at once from the Cauchy integral theorem. Since $\delta$ is arbitrary $R(u)$ does not depend on $c$ for $\sigma_1 < c < \sigma_2$.

Given $k$ in (3.5) it follows from (4.3) with $c = k + \delta_1$ and $c = k - \delta_1$, for some sufficiently small $\delta_1 > 0$, that (3.5) holds.

To prove (3.6) let

$$\begin{aligned} F(x) &= \int_0^{\infty} R(u) \frac{e^{-ux}}{1+e^{-ux}} du \\ &= \frac{1}{i(2\pi)^{1/2}} \int_0^{\infty} \frac{e^{-ux}}{1+e^{-ux}} du \int_{-i\infty+c}^{i\infty+c} \frac{\exp((s+ip)^2/2)u^{s-1}}{\Gamma(s)\zeta(s)(1-2^{1-s})} ds. \end{aligned}$$

Since the repeated integral is absolutely convergent the order may be inverted to give

$$F(x) = \frac{1}{i(2\pi)^{1/2}} \int_{-i\infty+c}^{i\infty+c} \frac{\exp((s+ip)^2/2)}{\Gamma(s)\zeta(s)(1-2^{1-s})} ds \int_0^\infty \frac{e^{-ux}}{1+e^{-ux}} u^{s-1} du.$$

Since

$$\int_0^\infty \frac{e^{-ux}}{1+e^{-ux}} u^{s-1}\, du = x^{-s}\Gamma(s)\zeta(s)(1-2^{1-s})$$

it follows that

$$F(x) = \frac{1}{i(2\pi)^{1/2}} \int_{-i\infty+c}^{i\infty+c} (\exp(s+ip)^2/2) x^{-s} ds.$$

Setting $s+ip=iw$ and using Cauchy's integral theorem

$$F(x) = \frac{x^{ip}}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-w^2/2} x^{-iw} dw$$

$$= \frac{x^{ip}}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-w^2/2} \exp(-iw \log x)\, dw = x^{ip} \exp(-\log^2 x/2)$$

which proves (3.6).

5. If (1.2) implies that $f(x)$ is zero then (1.1) is valid. Indeed (1.2) implies that any $g(x) \in L^2(0, \infty)$ can be approximated arbitrarily well in $L^2(0, \infty)$ by the functions

$$\frac{e^{-\lambda_n x}}{1+e^{-\lambda_n x}} (x^{\alpha-1/2} + x^{\beta-1/2}).$$

Thus given any $\epsilon$ there exist $N$ and $a_n$, $1 \leqq n \leqq N$, such that

$$\int_0^\infty \left| g(x) - \sum_1^N a_n \frac{e^{-\lambda_n x}}{1+e^{-\lambda_n x}} (x^{\alpha-1/2} + x^{\beta-1/2}) \right|^2 dx < \epsilon.$$

Let

$$g(x) = e^{-x}(x^{\alpha-1/2} + x^{\beta-1/2}).$$

Then

$$\int_0^\infty \left( e^{-x} - \sum_1^N a_n \frac{e^{-\lambda_n x}}{1+e^{-\lambda_n x}} \right)^2 (x^{2\alpha-1} + x^{2\beta-1}) \frac{(x^{\alpha-1/2} + x^{\beta-1/2})^2}{x^{2\alpha-1} + x^{2\beta-1}} < \epsilon.$$

Since the numerator of the last term exceeds the denominator (1.1) follows.

Thus it is seen that if (1.2) implies $f(x)$ is zero then (1.1) holds. This in turn implies $\zeta(s)$ has no zeros in the strip $(\sigma_1, \sigma_2)$ which proves the sufficiency of the condition of Theorem II and completes the proof of Theorem II.

If $\zeta(s)$ has no zeros in the strip $(\sigma_1, \sigma_2)$ then (1.2) implies $f(x)$ is zero which in turn implies that (1.1) holds. Thus (1.1) is a necessary condition and this completes the proof of Theorem I.

To prove the remark at the end of Theorem I note that the closure property of the translations in $L^2(-\infty, \infty)$ of Wiener [7] shows that the functions

$$x^{c-1/2} \frac{e^{-ax}}{1 + e^{-ax}} \qquad (0 < a < \infty)$$

where $c$ is fixed, $1/2 \leqq c \leqq 1$, are closed in $L^2(0, \infty)$. Using the result of §3 based on Carleman's theorem it follows easily that

$$x^{c-1/2} \frac{e^{-\lambda_n x}}{1 + e^{-\lambda_n x}} \qquad (n = 1, 2, \cdots)$$

are closed in $L^2(0, \infty)$. Thus if $g(x) \in L^2(0, \infty)$, then given any $\epsilon > 0$ there exists $N$ and $\{a_n\}$ such that

$$\int_0^\infty \left( \sum_1^N a_n x^{c-1/2} \frac{e^{-\lambda_n x}}{1 + e^{-\lambda_n x}} - g(x) \right)^2 dx < \epsilon.$$

Letting $g(x) = x^{c-1/2} e^{-x}$ the remark is proved.

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Massachusetts Institute of Technology