BOUNDARY VALUES OF CONTINUOUS ANALYTIC FUNCTIONS

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Let U , K , and C denote the open unit disc, the closed unit disc, and the unit circumference, respectively. Let A be the set of all complex-valued functions which are defined and continuous on K and analytic in U .

The theorem proved in this paper states that if E is a closed subset of C , of (Lebesgue) measure zero, and if ϕ is a continuous function defined on E , then there exists a function $f \in A$ which is an extension of ϕ , i.e., $f(z) = \phi(z)$ for all $z \in E$ (actually, a little more is proved about the way in which the range of f can be restricted). This theorem may be regarded as a strengthened form of a result due to Fatou [3, p. 393] which asserts that under the above assumptions on E there exists a function $f \in A$ which vanishes at every point of E and at no other point of K .

A well-known result due to F. and M. Riesz [4] shows that the theorem is false for any set E whose closure has positive measure. It is perhaps more interesting to observe that the conclusion of the theorem cannot be strengthened by asserting, for instance, that f satisfies a Lipschitz condition if ϕ does: if E consists of the points 1 and $\exp(i/\log n)$ ($n = 2, 3, 4, \dots$), and if $\phi(z) = 1/z$, then for any function $f \in A$ which extends ϕ , the function $g(z) = zf(z)$ is nonconstant ($g(0) = 0$) and $g(z) = 1$ for all $z \in E$; this function g cannot satisfy a Lipschitz condition [1, p. 13] and the same is true of f . Carleson's work [2] on this and related problems should be mentioned here.

A subset of the plane homeomorphic to K will be called a two-cell. For every two-cell T there exists a mapping which will be called a conformal mapping of K onto T ; more precisely, this is a homeomorphism of K onto T which maps U conformally onto the interior of T .

The square with vertices at 0, 1, i will be denoted by S . For any real number t , tS is the set of all z of the form $z = tw$, $w \in S$.

A function is said to be simple if it has only a finite set of values.

THEOREM. *Suppose*

- (a) E is a closed subset of C , $m(E) = 0$;
- (b) ϕ is a continuous complex valued function on E ;

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(c) T is a two-cell such that $\phi(E) \subset T$.

Then there exists a function $f \in A$ such that

(i) $f(z) = \phi(z)$ for all $z \in E$;

(ii) $f(K) \subset T$.

LEMMA 1. If ϕ is a simple continuous function on E whose real part is non-negative, then there exists a function $f \in A$ such that (i) f is an extension of ϕ , and (ii) $R[f(z)] \geq 0$ for all $z \in K$.

PROOF. Since every simple continuous function is a finite linear combination of continuous characteristic functions, it suffices to consider the case in which $E = E_0 \cup E_1$ and $\phi(z) = 0$ on E_0 , $\phi(z) = \alpha$ on E_1 , with $R[\alpha] \geq 0$, $\alpha \neq 0$.

For every closed set $H \subset C$ of measure zero, one can construct an integrable function $\mu > 1$ on C such that

(a) μ is continuous on $C - H$,

(b) if $z_0 \in H$, $\mu(z_0) = +\infty$ and $\mu(z) \rightarrow +\infty$ as $z \rightarrow z_0$,

(c) on any closed subarc of $C - H$, μ has a bounded derivative.

Properties (a) and (b) can be summarized by saying that μ is continuous on C , in the extended sense.

The Poisson integral of this function μ yields a function $u(H, z)$, defined on K , which is continuous on K in the extended sense, and which has the value $+\infty$ at every point of H (and nowhere else); the conjugate harmonic function $v(H, z)$ is continuous on $K - H$ [3, pp. 342-344, 360-361]. Put

$$g(H, z) = u(H, z) + iv(H, z) \quad (z \in K)$$

with the understanding that $g(H, z) = \infty$ if $z \in H$. The values of g lie in the half-plane $u > 1$, so that g has a single-valued square root $h(H, z)$, defined on K , whose values lie in the domain $|\arg h| < \pi/4$, $R[h] > 1$; note that $h(H, z) = \infty$ if and only if $z \in H$.

It is now easy to verify that the function

$$q(z) = \frac{h(E_1, z)}{h(E_0, z) + h(E_1, z)} \quad (z \in K)$$

is a member of A , that $q(z) = 0$ if and only if $z \in E_0$, $q(z) = 1$ if and only if $z \in E_1$, and that $0 \leq R[q(z)] \leq 1$ for all $z \in K$. Thus $q(K)$ is contained in a closed rectangle M which lies in the closed right half-plane and has 0 and 1 as boundary points. Let M_1 be another such rectangle, with 0 and α as boundary points, and let ψ be a conformal mapping of M onto M_1 which satisfies the conditions $\psi(0) = 0$, $\psi(1) = \alpha$. Then $f(z) = \psi(q(z))$ is the desired function.

LEMMA 2. If ϕ is a simple continuous function on E which maps E

into a two-cell T , then there exists a function $f \in A$ which is an extension of ϕ and which maps K into T .

PROOF. Let z_0 be a boundary point of T which is not a value of ϕ , and let ψ map the right half-plane conformally onto the interior of T , such that $\psi(\infty) = z_0$. By Lemma 1, there is a function $g \in A$ with non-negative real part, which coincides with $\psi^{-1}(\phi(z))$ for all $z \in E$; thus the function $f(z) = \psi(g(z))$ has the desired properties.

LEMMA 3. If ϕ is a continuous function on E which maps E into S , then there exists a sequence $\{\phi_n\}$ of simple continuous functions on E such that

$$(i) \quad \phi(z) = \sum_{n=1}^{\infty} \phi_n(z) \quad (z \in E),$$

$$(ii) \quad \phi_n(E) \subset 2^{-n}S \quad (n = 1, 2, 3, \dots).$$

PROOF. Let $\phi_0(z) = 0$ and suppose $\phi_0, \dots, \phi_{n-1}$ have been defined for some $n \geq 1$, such that

$$\lambda_{n-1}(E) \subset 2^{-n+1}S,$$

where $\lambda_{n-1} = \phi - \phi_0 - \dots - \phi_{n-1}$. Since E is totally disconnected, E is the union of disjoint closed sets $E_1, \dots, E_p$ such that the oscillation of λ_{n-1} on E_k is less than 2^{-n} . That is to say, there are squares $Q_1, \dots, Q_p$ with edges of length 2^{-n} and parallel to the axes, such that

$$\lambda_{n-1}(E_k) \subset Q_k \subset 2^{-n+1}S \quad (1 \leq k \leq p).$$

It is clear that Q_k and $2^{-n}S$ have a point c_k in common. Define a function ϕ_n on E by

$$\phi_n(z) = c_k \quad (z \in E_k, 1 \leq k \leq p).$$

If $\lambda_n = \lambda_{n-1} - \phi_n = \phi - \phi_0 - \dots - \phi_n$, then $\lambda_n(E) \subset 2^{-n}S$, and the process continues. This proves the lemma.

PROOF OF THE THEOREM. Suppose first that $T = S$, and let ϕ_n be the functions of Lemma 3. By Lemma 2 there are functions $g_n \in A$ which are extensions of ϕ_n and which map K into $2^{-n}S$ ($n = 1, 2, 3, \dots$). Define

$$f(z) = \sum_{n=1}^{\infty} g_n(z) \quad (z \in K).$$

The fact that $g_n(K) \subset 2^{-n}S$ for $n = 1, 2, 3, \dots$ implies first of all that the series converges uniformly, so that $f \in A$, and secondly that $f(K) \subset S$; for $z \in E$,

$$f(z) = \sum_{n=1}^{\infty} g_n(z) = \sum_{n=1}^{\infty} \phi_n(z) = \phi(z).$$

The general case follows by an application of the Riemann mapping theorem, as in the proof of Lemma 2.

AN APPLICATION (*Added* July 31, 1956). An algebra R of continuous complex-valued functions on K is called a *maximum modulus algebra* if to every $f \in R$ there is a point $z_0 \in C$ such that $\max_{z \in K} |f(z)| = |f(z_0)|$. In [5] the following is proved: If R is a maximum modulus algebra which contains (i) a function which is one-to-one, (ii) a non-constant function which is analytic in U , then every member of R is analytic in U . It is now possible to show that (i) cannot be omitted from the hypotheses.

To do this, consider the bicylinder $K \times K$ (in the space of two complex variables) whose distinguished boundary is the Cartesian product $C \times C$. Let E be a perfect subset of C , of measure zero. There exist continuous functions ϕ and ψ , defined on E , such that the mapping

$$z \rightarrow (\phi(z), \psi(z))$$

maps E onto $C \times C$. The theorem proved in the present paper shows that there is a function f , continuous on K , analytic in U , such that $f(K) \subset K$ and $f(z) = \phi(z)$ on E . Let g be a function which is continuous on K , *not* analytic in U , such that $g(K) \subset K$ and $g(z) = \psi(z)$ on E ; we could take g to be conjugate-analytic in U .

Since every function analytic on $K \times K$ attains its maximum modulus on $C \times C$, it is clear that every member of the algebra generated by f and g attains its maximum modulus on E . We have thus constructed a maximum modulus algebra which contains both analytic and nonanalytic functions; it is to be noted, however, that this algebra does not separate points on K .

REFERENCES

1. Arne Beurling, *Ensembles exceptionnels*, Acta Math. vol. 72 (1940) pp. 1-13.
2. Lennart Carleson, *Sets of uniqueness for functions regular in the unit circle*, Acta Math. vol. 87 (1952) pp. 325-345.
3. P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math. vol. 30 (1906) pp. 335-400.
4. F. and M. Riesz, *Über die Randwerte einer analytischen Funktion*, Quatrième Congrès des Math. Scand., 1916, pp. 27-44.
5. Walter Rudin, *Analyticity and the maximum modulus principle*, Duke Math. J. vol. 20 (1953) pp. 449-458.

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