References

1. I. Gelfand, *Normierte Ringe*, Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941) pp. 3–24.
2. I. Gelfand and M. Neumark, *On the embedding of normed rings into the ring of operators in Hilbert Space*, Rec. Math. (Mat. Sbornik) N.S. vol. 12 (1943) pp. 197–213.
3. L. J. Heider, *A note on a theorem of K. G. Wolfson*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 305–308.
4. N. Jacobson, *The theory of rings*, Mathematical Surveys, vol. 2, New York, 1943.
5. K. G. Wolfson, *The algebra of bounded functions*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 10–14.

Rutgers University

LIE SIMPLICITY OF A SPECIAL CLASS OF ASSOCIATIVE RINGS[1]

WILLARD E. BAXTER

Given an associative ring A, by introducing a new multiplication we can form from it a new ring called the Lie ring of A. This multiplication is defined by $[a, b]=ab-ba$ for all $a, b\in A$. If U is an additive subgroup of A and if for arbitrary $u\in U$, $a\in A$, $ua-au\in U$, then U is said to be a Lie ideal of A. If X, Y are additive subgroups of A then by $[X, Y]$ we mean the additive subgroup generated by all the elements $xy-yx$, where $x\in X$, $y\in Y$. An additive subgroup U of $[A, A]$ is said to be a proper Lie ideal of $[A, A]$ if $U\neq[A, A]$ and if $[U, [A,A]]\subset U$.

In [4], Herstein proved that if A is a simple ring of characteristic not 2 or 3, and if U is a proper Lie ideal of $[A, A]$, then U is contained in Z, the center of A. In this paper we settle the question in the open case where A is a simple ring of characteristic 2 or 3. The above theorem becomes sharpened to:

Theorem 1. *If A is a simple ring and if U is a proper Lie ideal of $[A, A]$, then U is contained in the center of A, except for the case where A is of characteristic 2 and 4 dimensional over its center, a field of characteristic 2.*

Presented to the Society, October 22, 1955; received by the editors September 26, 1955.

[1] The results of this paper will comprise the beginning portion of a thesis, which will be presented to the Faculty of the Graduate School of the University of Pennsylvania in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

We settle each of the cases, characteristic 2 and characteristic 3, separately.

LEMMA 1. *If A is a simple ring of characteristic* 2, *and if U is a Lie ideal of $[A, A]$ such that $[U, U]\subset Z$, the center of A, then $U\subset Z$ except in the case that A is a simple algebra of dimension* 4 *over its center.*

PROOF. If $a\in U$; $a\notin Z$ and $x\in[A, A]$ then $ax+xa\in U$ since U is a Lie ideal of $[A, A]$. By assumption, $[U, U]\subset Z$ and so $a(ax+xa)+(ax+xa)a\in Z$; that is, $a^2x+xa^2\in Z$ for all $x\in[A, A]$. Now, let $T=\{t\in A\mid [t, [A, A]]\subset Z\}$. We note that $a^2\in T$. We claim that T is a Lie ideal of A. For if $t\in T$, $x\in A$, $y\in[A, A]$ then

$$(tx + xt)y + y(tx + xt) = \{t(xy + yx) + (xy + yx)t\} + \{x(ty + yt) + (ty + yt)x\}.$$

Since $xy+yx\in[A, A]$ then $t(xy+yx)+(xy+yx)t\in Z$; also since $t\in T$, $y\in[A, A]$ then $ty+yt\in Z$ whence $x(ty+yt)+(ty+yt)x=0$. Thus $[tx+xt, [A, A]]\subset Z$ and $tx+xt\in T$. So T is a Lie ideal of A, consequently, by a result of Herstein [2], except where A is the exceptional case noted in the statement of the theorem, $T\subset Z$ or $T\supset[A, A]$. In either case this places a^2 in the center. If, on the one hand, $T\subset Z$, then since $a^2\in T$, $a^2\in Z$ results. If, on the other hand, $T\supset[A, A]$, then the fact that $a\in U\in[A, A]$ forces $[a, [A, A]]\subset Z$; in particular $c=a(ax+xa)+(ax+xa)a$ is in Z for all $x\in A$. Substituting ax for x in this we obtain $ac\in Z$. Since the center of a simple ring is a field, if $c\neq 0$, then $c^{-1}\in Z$ and thus $a=ac\cdot c^{-1}\in Z$ which is a contradiction. This forces $c=0$. That is, $a(ax+xa)=(ax+xa)a$, which implies that $a^2x=xa^2$ for all $x\in A$. Thus $a^2\in Z$. If $a\in U\cap Z$, of course, $a^2\in Z$. Hence for all $a\in U$, $a^2\in Z$.

If $a\in U$, x, $y\in A$ then $a(xy+yx)+(xy+yx)a\in U$. In this relation, replace y by ax and remembering that $a^2\in Z$ we obtain

$$(ax + xa)^2 \in U \qquad \text{for all } x \in A$$

and so by the above discussion

$$(ax + xa)^4 \in Z \qquad \text{for all } x \in A.$$

From this point the proof follows the same pattern as in the paper of Herstein [2] with only minor changes.

We note that should $Z=(0)$, then $a^2=0$ and also $(ax+xa)^4=0$ for all $x\in A$. This forces $(ax)^5=0$ for all $x\in A$; that is, aA is a nil-right ideal of bounded index 5, which is impossible (cf. Herstein [2, proof of Theorem 2]) in a simple ring.

Thus, we can assume that $Z\neq(0)$ and likewise, suppose $a^2\neq 0$ for some $a \in U$. Since Z is a field of characteristic 2 and nontrivial, A is a primitive ring, consequently, it is a dense ring of linear transformations acting on a vector space V over a division ring D. We wish to show that V is 2 dimensional over D and that $D = Z$. This leads to establishing our theorem.

By extending the center we may, without loss of generality, assume that there exist an $a \in U$, $a \notin Z$ such that $a^2 = 1$ and $(ax + xa)^4 \in Z$ for all $x \in A$. (cf. Herstein [2].) Since $(a+1)^2 = 0$ and $a + 1 \neq 0$, A has zero divisors and so is not a division ring; being primitive it is a dense ring of linear transformations acting on a vector space V over a division ring D.

Almost as in Herstein [2], it follows that V is exactly 2 dimensional over D and that A is exactly the totality of all 2×2 matrices over D. All that remains to show is that $D = Z$, or in other words, that D is commutative.

We know that there exists a in A, a not in Z, such that a^2 is the identity matrix and $(ax + xa)^4$ is in Z for all x in A. Let

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix};$$

$\alpha, \beta, \gamma, \delta \in D$. Then

$$a^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which yields the following identities:

$$\alpha^2 + \beta\gamma = 1; \quad \alpha\beta + \beta\delta = 0; \quad \gamma\alpha + \delta\gamma = 0; \quad \gamma\beta + \delta^2 = 1.$$

Suppose that $\beta = 0$. The identities above reduce to $\alpha^2 = 1$, $\delta^2 = 1$, and so $\alpha = \delta = 1$, since D is a division ring of characteristic 2. Thus,

$$a = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}.$$

Since $(ax + xa)^4 \in Z$ for all $x \in A$, then in partidular this is true for

$$x = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \qquad \text{where } r \in D.$$

But

$$(ax + xa) = \begin{pmatrix} r\gamma & 0 \\ 0 & \gamma r \end{pmatrix},$$

and therefore

$$(ax + xa)^4 = \begin{pmatrix} (r\gamma)^4 & 0 \\ 0 & (\gamma r)^4 \end{pmatrix}$$

is in Z. This implies that $(r\gamma)^4$ is in Z for all $r \in D$. If $\gamma \neq 0$, then $r\gamma$ runs through all elements of D as r does, thus $s^4 \in Z$ for all $s \in D$. But then D is purely inseparable over its center which contradicts the fact that a noncommutative division ring contains at least one element which is separable over its center (Jacobson [5, Lemma 2]). Consequently $D = Z$ and our argument would be complete. So we suppose that $\gamma = 0$, then

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which places a in Z contrary to assumption.

If $\gamma = 0$ then

$$a = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

and if one lets

$$x = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}$$

the above argument carries through to show $D = Z$.

Now, suppose neither β nor γ are zero. Thus,

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

and letting

$$t = \begin{pmatrix} \delta\beta^{-1} + \beta^{-1} & 1 \\ \delta\beta^{-1} & 1 \end{pmatrix}, \quad t^{-1} = \begin{pmatrix} \beta & \beta \\ \delta & 1 + \delta \end{pmatrix},$$

then tat^{-1} is of form

$$\begin{pmatrix} r & 0 \\ s & p \end{pmatrix}.$$

Now, if $b = tat^{-1}$, then $b^2 \in Z$, $b \notin Z$, and $(bx + xb)^4 \in Z$ for all $x \in A$. Also

$$b = \begin{pmatrix} \alpha' & 0 \\ \gamma' & \delta' \end{pmatrix}$$

so by the above, if $D \neq Z$ then $b=1$, and so $a=1$ contrary to assumption. So if $U \not\subset Z$, $D=Z$, and so A is the set of all 2×2 matrices over Z, a field of characteristic 2, proving the lemma.

In [**4**], Herstein proved the following:

LEMMA 2. *If A is a simple ring of characteristic not 2, and if U is a Lie ideal of $[A, A]$; $U \neq [A, A]$ then $U^{(3)} = [[[U, U], [U, U]], [[U, U], [U, U]]] = (0)$.*

Although the condition that A not have characteristic 2 is imposed in the statement of his theorem, nowhere in his proof did Herstein make use of this fact. Thus by the proof in [**4**], Lemma 2 is true even when the characteristic is 2 and we shall use it in that case. The proof of Theorem 1 in the case of characteristic 2 follows.

Let U be a proper Lie ideal of $[A, A]$, then by Lemma 2, $U^{(3)} = (0)$. However, by definition, $U^{(3)} = [U^{(2)}, U^{(2)}]$ and so by Lemma 1, $U^{(2)} \subset Z$. $U^{(2)} = [[U, U], [U, U]]$ and hence again by Lemma 1, $[U, U] \subset Z$; and thus in the same manner $U \subset Z$, which is to be proved.

We now settle the case of characteristic 3. The argument follows that used by Herstein in [**3**].

LEMMA 3. *If A is a simple ring of characteristic 3 and U is Lie ideal of $[A, A]$ such that $[U, U] \subset Z$, the center of A, then $U \subset Z$.*

PROOF. Following the argument used in case 2 of Theorem 2 of Herstein [**3**], we can show that if a and b are in U and ab is also in U and if $[a, U] \neq 0$, then $[b, U]=0$. However, the set $T = \{t \in U \mid [t, U] = 0\}$ is a Lie ideal of $[A, A]$; $T \subset U$ and thus $[T, T]=0$. Therefore, by Herstein [**4**], $T \subset Z$. Since $b \in T$ we have $b \in Z$. Suppose now that $a \in U$, $a \notin Z$. For any $x \in A$, $b = a(ax-xa)-(ax-xa)a \in U$ since U is a Lie ideal of $[A, A]$. In this, replace x by ax and we get that $ab \in U$. Thus, by our previous discussion since $a, b \in U$, $a \notin Z$, and $ab \in U$ then $b \in Z$. Thus $b = a(ax-xa)-(ax-xa)a$ is in Z for all $x \in A$. Replace in this x by ax and we see that $ab \in Z$. Now, since $b \in Z$ if $b \neq 0$ then since Z is a field, $b^{-1} \in Z$ thus giving $a \in Z$ for all $a \in U$ which is the desired result. Thus, $b=0$; that is, $0 = a(ax-xa)-(ax-xa)a$ for all $x \in A$. If $[U, U]=0$ then the lemma follows from the results of Herstein [**4**]; thus we assume that there exist an $a \in U$, $u \in U$ such that $au-ua \neq 0$. In the above relation, $a(ax-xa)-(ax-xa)a=0$ replace x by ux. We then obtain

$$a(aux - uxa) = (aux - uxa)a.$$

That is: $a\{(au-ua)x+u(ax-xa)\} = \{(au-ua)x+u(ax-xa)\}a.$

Since a commutes with $ax-xa$ and $au-ua \in Z$ this relation yields $2(au-ua)(ax-xa)=0$. Since $2(au-ua) \neq 0$ by assumption and being in Z we must have $ax-xa=0$ for all x, which is impossible since it fails to be true for $x=u$. Thus $[U, U]=0$, and by Herstein [4] the lemma is proved. Thus, by the use of Lemma 2, Theorem 1 follows in the same way as it did in the case of characteristic 2. This settles the question for characreristic 3. Thus Theorem 1 is established.

Application. In his paper [4], I. N. Herstein conjectures that if A is a simple associative ring, then the proper subrings which are invariant under all automorphisms are contained in the center. He expressed a hope that the method of Lie ideals should be the approach taken in order to settle this question. Here, we illustrate how these methods do settle the question for a class of simple rings; this leads to a result of Hattori [1] and further extends a result of Iwahori [1] on invariant subspaces in certain simple rings. It is hoped that these methods can be extended to settle Herstein's conjecture in the general case. The following proof is due to I. N. Herstein.

Let A be an algebra over a field F, then a subspace U of A is said to be invariant if U is carried into itself by all inner automorphisms of A. Hattori reported that Iwahori [1] has proved that if A is a central simple algebra over a field F of characteristic 0, with descending chain conditions on left ideals, then the only invariant subspaces of A are (0), F, A, and $[A, A]$. We prove a generalization of this.

THEOREM 2. *If A is a central simple algebra, not a division algebra, over a field $F \neq G.F.$ (2) with descending chain conditions on left ideals, then the only invariant subspaces of A are* (0), F, A, $[A, A]$, *and possibly subspaces containing* $[A, A]$, *with the exception of the case in which A is the* 2×2 *matrices over* $G.F.$ (2).

PROOF. By the Wedderburn Theorem, $A=D_n$, D a division algebra. Since A is not a division algebra, n must be greater than or equal to 2. Let T be an invariant subspace of A and let $u \in A$ be such that $u^2=0$. Thus $(1+u)(1-u)=1$. Hence, if $t \in T$ then $(1+u)t(1-u) \in T$ since T is invariant. That is, $t+ut-tu-utu \in T$, which, since $t \in T$ means that

$$\text{(I)} \qquad ut - tu - utu \notin T.$$

If $u \in A$ is such that $u^2=0$ and if $\alpha \neq 0 \in F$, center of A, then $(\alpha u)^2=0$ and so since (I) holds for αu, we have $(\alpha u)t-t(\alpha u)-(\alpha u)t(\alpha u) \in T$. However, since $\alpha \neq 0 \in F$ and since T is a subspace, this gives

$$\text{(II)} \qquad ut - tu - \alpha utu \in T.$$

Multiplying (I) by α and subtracting from (II), we obtain $(1-\alpha)\cdot(ut-tu)\in T$. Now, since $F\neq$G.F. (2), there exist $\alpha\neq 0\in F$ such that $\alpha\neq 1$. Hence since T is a subspace, $ut-tu\in T$. By the Wedderburn Theorem we know that $A=D_n$, the set of all $n\times n$ matrices over a division ring D. The e_{ij}, for $i\neq j$, have the property that $(e_{ij})^2=0$ and so $e_{ij}t-te_{ij}\in T$. If $\delta\in D$, $(\delta e_{ij})^2=0$, for $i\neq j$, and therefore

$$\text{(III)} \qquad \delta e_{ij}t - t\delta e_{ij} \in T.$$

Now $t'=e_{ij}t-te_{ij}\in T$, for $i\neq j$, and hence, for all $\delta\in D$, $\delta e_{ji}t'-t'\delta e_{ji}\in T$. That is, $\delta e_{ji}(e_{ij}t-te_{ij})-(e_{ij}t-te_{ij})\delta e_{ji}\in T$. By the Jacobi identity, we have $\{e_{ij}[(\delta e_{ji})t-t(\delta e_{ji})]-[(\delta e_{ji})t-t(\delta e_{ji})]e_{ij}\}=\{t[e_{ij}(\delta e_{ji})-(\delta e_{ji})e_{ij}]-[e_{ij}(\delta e_{ji})-(\delta e_{ji})e_{ij}]t\}\in T$. However, by the above $e_{ij}[(\delta e_{ji})t-t(\delta e_{ji})]-[(\delta e_{ji})t-t(\delta e_{ji})]e_{ij}\}\in T$, and hence $t[e_{ij}(\delta e_{ji})-(\delta e_{ji})e_{ij}]-[e_{ij}(\delta e_{ji})-(\delta e_{ji})e_{ij}]t\in T$; that is:

$$\text{(IV)} \qquad t\delta(e_{jj}-e_{ii}) - \delta(e_{jj}-e_{ii})t \in T.$$

Since e_{ij}, $i\neq j$, and $e_{jj}-e_{ii}$ constitute a basis of $[A, A]$ over D, then $[T, [A, A]]\subset T$ by (III) and (IV); also $[T, [A, A]]\subset[A, A]$. If $T\supset[A, A]$, then the assertion of the theorem is correct and there is nothing to prove; so suppose $T\not\supset[A, A]$; it remains to show $T\subset F$. By the above, $T\cap[A, A]$ is a proper Lie ideal of $[A, A]$, therefore by Theorem 1, $T\cap[A, A]\subset F$, except when A is the set of all 2×2 matrices over a field of characteristic 2, which we investigate later. Thus, $[T, [A,A]]\subset T\cap[A,A]\subset F$. Let $W=\{x\in A \mid [x, [A,A]]\subset F\}$; clearly $T\subset W$. As before, W is a Lie ideal of A. Therefore, either $W\subset F$ or $W\supset[A, A]$. If $W\subset F$, then $T\subset W\subset F$. If, on the other hand, $W\supset[A, A]$, then $[[A, A], [A, A]]\subset F$, and by Lemmas 1 and 3 of Herstein [4] we have $[A, A]\subset F$, and so $A\subset F$ which is a contradiction. This then completes the proof where A is not the set of 2×2 matrices over a field of characteristic 2. Suppose therefore that F is a field of characteristic 2, not G.F. (2), and that A is the set of 2×2 matrices over this field. Let

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in T,$$

with the condition that $\gamma\neq 0$. Then by combining (I) and (II) in the above argument

$$e_{12}ae_{12} = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} \in T$$

and thus $e_{12}\in T$. Let

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

then $xe_{12}x^{-1} = e_{21} \in T$, since T is invariant. Let

$$y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix};$$

then $ye_{12}y^{-1} = e_{11} + e_{12} + e_{21} + e_{22} \in T$ and therefore $e_{11} + e_{22} \in T$ and, consequently, $[A, A] \subset T$. Now, if $\gamma = 0$ and $\beta \neq 0$ then the same argument follows by using e_{21} instead. Thus, assume $\beta = \gamma = 0$, then

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}.$$

Now, if $\alpha = \delta$ for all $a \in T$ then $T \subset F$; and the theorem is proved. So assume that there exist α and δ such that $\alpha \neq \delta$. Let

$$y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};$$

then

$$yay^{-1} = \begin{pmatrix} \alpha & \alpha + \delta \\ 0 & \delta \end{pmatrix} \in T$$

and since $\alpha + \delta \neq 0$ we are in the situation $\beta \neq 0$ of above. This then completes the proof.

LEMMA 4. *If A is a simple ring not a field and T is a subring of A which contains $[A, A]$, then $T = A$.*

PROOF. T is a Lie ring of A since $[T, A] \subset [A, A] \subset T$. Thus T is both a subring and a Lie ring of A. If A is not both of characteristic 2 and 4 dimensional over its center, a field of characteristic 2, then $T = A$ (cf. Herstein [2]). Thus, the theorem remains to be proved in the case where A is of characteristic 2 and 4 dimensional over its center F, a field of characteristic 2. If A is the set of 2×2 matrices over F then since $T \supset [A, A]$, T contains e_{12} and e_{21}. However, T is a subring and therefore T also contains $e_{12}e_{21} = e_{11}$ and $e_{21}e_{12} = e_{22}$. Thus $T = A$. Lastly, let A be a 4 dimensional division algebra of characteristic 2. Let $t_1, t_2 \in T$ and $a \in A$, then $t_1(t_2a) - (t_2a)t_1 \in [A, A] \subset T$. However, $t_1(t_2a) - (t_2a)t_1 = (t_1t_2 - t_2t_1)a + t_2(t_1a - at_1) \in T$. Since $at_1 - t_1a \in [A, A] \subset T$, and since $t_2 \in T$ it follows that $t_2(t_1a - at_1) \in T$, so we have $(t_1t_2 - t_2t_1)a \in T$. Hence either $t_1t_2 - t_2t_1 = 0$ for all $t_1, t_2 \in T$ or $A = T$. If $[T, T] = 0$, then $[[A, A], [A, A]] = 0$ and as in Herstein

[4], we have $A \subset F$, which is a contradiction. Thus, the lemma is proved.

We are also able to prove Hattori's theorem [1] about invariant subalgebras.

THEOREM 3. *Let A be a central simple, finite dimensional algebra over a field F. If T is a subalgebra invariant under every inner automorphism of A, then either $T=A$ or $T\subset F$, except in the case in which A is a total matrix ring of degree* 2 *over G.F.* (2).

PROOF. If A is a division algebra, then T is a division subalgebra and so the theorem follows from the Cartan-Hua-Brauer Theorem. If A is not a division algebra then by Theorem 2 either $T\supset [A, A]$ or $T\subset F$. By Lemma 4, if $T\supset [A, A]$ then $T=A$.

The counter example in the case where A is a total matrix ring of degree 2 over G.F. (2) is given in Hattori [1].

Added in revision. In a forthcoming paper S. A. Amitsur, independently, obtains these results.

REFERENCES

1. A. Hattori, *On invariant subrings*, Jap. J. Math. vol. 21 (1951) pp. 121–129.

2. I. N. Herstein, *On the Lie and Jordan rings of a simple associative ring*, Amer. J. Math. vol. 77 (1955) pp. 393–403.

3. ———, *On the Lie ring of a division ring*, Ann. of Math. vol. 60 (1954) pp. 571–575.

4. ———, *On the Lie ring of a simple associative ring*, Duke Math. J. vol. 22 (1955) pp. 471–476.

5. N. Jacobson, *Structure theory for algebraic algebras of bounded degree*, Ann. of Math. vol. 46 (1945) pp. 695–707.

6. J. Levitzki, *A problem of A. Kurosch*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 1033–1035.

UNIVERSITY OF PENNSYLVANIA