3. E. Helly, *Über lineare funktional operationen*, Sitzungsberichte der Mathematisch-Naturwissenschaftlichen Klasse der Kaiserlichen Akademie der Wissenschaften vol. 121 (1912) pp. 265–297.

4. E. W. Hobson, *The theory of functions of a real variable and the theory of Fourier's series*, 2d ed., Cambridge, Cambridge University Press, 1921.

5. J. A. Shohat and J. D. Tamarkin, *The problem of moments*, Mathematical Surveys no. 14, New York, 1943.

6. D. V. Widder, *The Laplace transform*, Princeton University Press, 1941.

7. W. H. Young, *On multiple integrals*, Proc. Roy. Soc. London Ser. A vol. 93 (1916) pp. 28–41.

UNIVERSITY OF MISSOURI

ABELIAN RINGS AND SPECTRA OF OPERATORS ON l_p

G. L. KRABBE

1. **Introduction.** Let l_p denote the set of all sequences $c=\{c_n\}$ such that $\|c\|_p=(\sum_{n=-\infty}^{\infty}|c_n|^p)^{1/p}<\infty$. If a and c are sequences, the convolution $a*c$ is defined as the sequence $\{b_n\}$ such that

$$[a*c]_n = b_n = \sum_{\nu=-\infty}^{\infty} a_{n-\nu}c_\nu \qquad (n = 0, \pm 1, \pm 2, \pm 3, \cdots).$$

Suppose A is a bounded and summable function on the interval $[-\pi, \pi]$, and denote by ΦA the sequence $\{a_n\}$ of Fourier coefficients of A. The "Laurent matrix" $(a_{n-\nu})$ represents a transformation $A_{\#p}$ defined for any c in l_p by $A_{\#p}c=(\Phi A)*c$. The following properties were proved for $p=2$ by O. Toeplitz [11] and F. Riesz [8, pp. 171–175]:

(i) *$A_{\#p}$ is a bounded operator on l_p,*

(ii) *if $B(\theta)=[A(\theta)]^{-1}$ defines a bounded function B, then the inverse $(A_{\#p})^{-1}$ of $A_{\#p}$ exists, and $(A_{\#p})^{-1}=B_{\#p}$,*

(iii) *if A is continuous, then the spectrum $\sigma(A_{\#p})$ of $A_{\#p}$ is the range*[1] *of A.*

Assume henceforth that $p>1$. This case was not considered by Toeplitz and Riesz; their results depend on the circumstance[2] that

Presented to the Society April 15, 1955 under the titles *Certain abelian rings of operators on l_p* and *On the spectra of certain operators on l_p*; received by the editors January 31, 1955 and, in revised form, July 25, 1955 and October 11, 1955.

[1] The range of A is the image $A([-\pi, \pi])$.

[2] The introductions to [3; 4] contain a concise account of these matters.

Φ is an isometric mapping of $L^p([-\pi, \pi])$, $p=2$, onto some l_q, which no longer holds when $p\neq 2$. S. B. Stečkin [10] recently established (i) by restricting A to the set $\mathfrak{B}$ of functions of bounded variation on $[-\pi, \pi]$.

Suppose $A\in\mathfrak{B}$. The validity of (ii)–(iii) remains to be considered. In this paper, we verify (ii) and prove that, if A is continuous, then $\sigma(A_{\#p})$ is a connected subset of the range of A. The statement (iii) holds when A is in the set $\mathfrak{A}$ of functions analytic on $[-\pi, \pi]$. The proofs hinge on the fact that the set $\{A_{\#p}: A\in\mathfrak{B}\}$ forms an abelian ring isomorphic to $\mathfrak{B}$.

Let I be the identity function; we shall show that

(iv) *if* $A\in\mathfrak{A}$, *then* $A_{\#p}=A(I_{\#p})$ *and* $A_{\#p}$ *has no eigenvalues when* A *is not a constant function.*

Here $A(I_{\#p})$ is to be interpreted in the functionality sense of the Dunford operational calculus. The operator $I_{\#p}$ is represented by the matrix $(a_{n\nu})$, where $a_{n\nu}=i(-1)^{n+\nu}/(n-\nu)$, $a_{nn}=0$. From (iv) follows that $I_{\#p}$ has a purely continuous spectrum consisting of the whole interval $[-\pi, \pi]$.

1.1. *Application.* Suppose f belongs to the ring of all functions f whose Laurent series $\sum a_n\lambda^n$ converges absolutely on the circumference $\Gamma_1=\{\lambda: |\lambda|=1\}$. Take a fixed number p, $p>1$. If $f_\bigstar$ denotes the transformation defined by $f_\bigstar c=a*c$ $(c\in l_p)$, then $f_\bigstar$ is a bounded operator on l_p and the spectrum $\sigma(f_\bigstar)$ is the image $f(\Gamma_1)$ of Γ_1 by f. This generalizes results proved in the Hilbert space case $p=2$ by Toeplitz [**12**; **13**], and is easily derived from (i) and (iii) as follows. Set $A(\theta)=f(e^{i\theta})$; then $A(\theta)=\sum a_n e^{in\theta}$, which implies $a=\Phi A$, $f_\bigstar c=A_{\#p}c$, $f_\bigstar=A_{\#p}$, and $\sigma(f_\bigstar)=\sigma(A_{\#p})$. But $A\in\mathfrak{A}$ (since f is analytic on Γ_1); from (i) and (iii) we can now infer that $f_\bigstar$ is a bounded operator and $\sigma(A_{\#p})=A([-\pi, \pi])$. The observation $A([-\pi, \pi])=f(\Gamma_1)$ accordingly yields the conclusion $\sigma(f_\bigstar)=f(\Gamma_1)$.

2. **Preliminaries.** Denote by $\mathfrak{B}$ the set of all functions whose domains include the closed interval $[-\pi, \pi]$ and which are of bounded variation there. If $A\in\mathfrak{B}$, we define ΦA as the sequence a such that

$$(1)\qquad [\Phi A]_n = a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}A(\theta)d\theta \qquad (n = 0, \pm 1, \pm 2, \pm 3, \cdots).$$

Suppose p fixed throughout, and $p>1$. By Hölder's inequality $|[a*c]_n|\leqq\|a\|_q\|c\|_p$ for some $q>1$. Therefore $a*c$ is defined for any c in l_p, since $a=\Phi A$ implies $\|a\|_q<\infty$ for any $q>1$ [**2**, Theorem 37]. In consequence, the transformation $A_\#$ (which was denoted $A_{\#p}$ in the Introduction) can be defined by

$$A_{\#}c = (\Phi A) * c \qquad \text{for any } c \text{ in } l_p. \tag{2}$$

We denote by $\mathfrak{E}$ the ring of all bounded operators on l_p (an operator on l_p is a linear mapping of l_p into itself). Stečkin [10] has proved that

(v) *if* $A \in \mathfrak{B}$, *then* $A_{\#} \in \mathfrak{E}$.

2.1. Lemma. *Let S_0 be the set of all sequences c such that $c_n = 0$ for all $|n|$ sufficiently large. If T_1 and T_2 are members of $\mathfrak{E}$ satisfying $T_1c = T_2c$ for all c in S_0, then $T_1 = T_2$.*

Proof. It is easily shown that S_0 is dense in l_p. The operators T_1 and T_2, being in $\mathfrak{E}$, are continuous on l_p; since they coincide on S_0, the conclusion follows.

2.2. Theorem. *If A and B are in $\mathfrak{B}$, then $A_{\#}B_{\#} = (A \cdot B)_{\#}$, where $A \cdot B$ is the function C with $C(\theta) = A(\theta) \cdot B(\theta)$.*

Proof. We see from (v) that $T_1 = A_{\#}B_{\#}$ and $T_2 = (A \cdot B)_{\#}$ are members of $\mathfrak{E}$. In view of 2.1 and $S_0 \subset l_2$, it will therefore suffice to show that $(A_{\#}B_{\#})x = (A \cdot B)_{\#}x$ for all x in l_2. To that effect, we first recall that [2, (3.1.1)]

$$(\Phi F) * (\Phi G) = \Phi(F \cdot G) \qquad \text{when } F \in L^2 \text{ and } G \in L^2. \tag{3}$$

If $x \in l_2$, it follows from the Fischer-Riesz theorem that there exists a function X in L^2 with $x = \Phi X$. Now $\{A, B\} \subset \mathfrak{B} \subset L^2$ and $B \cdot X \in L^2$ (since B is bounded); keeping (2) in mind, we apply (3) twice in succession to obtain

$$(A_{\#}B_{\#})x = (\Phi A) * \{(\Phi B) * (\Phi X)\} = (\Phi A) * \{\Phi(B \cdot X)\} = \Phi(A \cdot [B \cdot X]).$$

This concludes the proof, since $(A \cdot B)_{\#}x = \{\Phi(A \cdot B)\} * (\Phi X) = \Phi([A \cdot B] \cdot X)$ follows again from (3) and $A \cdot B \in L^2$.

3. **The ring $\mathfrak{B}_{\#}$.** From (v) and 2.2 can easily be deduced that the set $\mathfrak{B}_{\#} = \{A_{\#}: A \in \mathfrak{B}\}$ forms an abelian subring of $\mathfrak{E}$. The unit element of the ring $\mathfrak{B}$ is the function I^0 such that $I^0(\theta) = 1$; note that $I^0_{\#} = 1$ (the identity operator). If $A \in \mathfrak{B}$, then A^{-1} is the function B defined by $B(\theta) = [A(\theta)]^{-1}$.

3.1. Remark. Suppose $A \in \mathfrak{B}$. If A^{-1} is a bounded function, it is easily seen that $B = A^{-1} \in \mathfrak{B}$, so that $(A_{\#})^{-1} = (A^{-1})_{\#} \in \mathfrak{B}_{\#}$ follows from 2.2 and $A_{\#}B_{\#} = I^0_{\#} = 1$.

3.2. Theorem. *The linear transformation $A \to A_{\#}$ is an isomorphic mapping of the ring $\mathfrak{B}$ onto the ring $\mathfrak{B}_{\#}$.*

Proof. In view of 2.2, it will be enough to show that the mapping $A \to A_{\#}$ is one-to-one. If $A_{\#} = B_{\#}$, we can infer from (2) that $(\Phi A) * c$

$=(\Phi B)*c$, where $c_0=1$ and $c_n=0$ if $n\neq 0$; but $a*c=a$, and therefore $\Phi A=\Phi B$. This implies that $A=B$ almost-everywhere, and the proof is complete.

4. **The subring of continuity.** Let $\mathfrak{K}$ denote the set of all members of $\mathfrak{B}$ which are continuous on $[-\pi, \pi]$. The sets $\mathfrak{K}$ and $\mathfrak{K}_\#=\{A_\#: A\in\mathfrak{K}\}$ are provided with the following respective norms; $\|A\|=\sup\{|A(\theta)|: |\theta|\leqq\pi\}$ when $A\in\mathfrak{K}$, and $\|T\|=\sup\{\|Tc\|_p: \|c\|_p\leqq 1\}$ when $T\in\mathfrak{K}_\#$. We recall that $Z=\lim X_s$ when $\lim\|Z-X_s\|=0$. If $\sigma(X)$ denotes the spectrum of an element X of a Banach algebra [5, p. 97], it is easily checked that $\sigma(A)$ is the range $A([-\pi, \pi])$ of A, whenever $A\in\mathfrak{K}$.

4.1. THEOREM. *The set $\mathfrak{K}_\#$ forms an irreducible ring, in the sense that* 1 *and* O *are the only members T of $\mathfrak{K}_\#$ such that $T^2=T$.*

PROOF. Suppose $T^2=T\in\mathfrak{K}_\#$. Then $T=A_\#=(A_\#)^2$, and a successive application of 2.2 and 3.2 yields $A_\#=(A^2)_\#$ and $A=A^2$. Now $\mathfrak{K}$ forms an irreducible ring [6, p. 417 (11)], and consequently $A=I^0$ or $A=O$; thus, either $T=I^0_\#=1$ or $T=O_\#=O$.

4.2. THEOREM. *If $C\in\mathfrak{K}$, then $\sigma(C_\#)$ is a connected subset of $\sigma(C)$. In case $\lambda\notin\sigma(C)$, then $(\lambda 1-C_\#)^{-1}=(A^{-1})_\#\in\mathfrak{K}_\#$ where $A=\lambda I^0-C$.*

PROOF. Suppose $\lambda\notin\sigma(C)$; then $\lambda-C(\theta)=A(\theta)\neq 0$ for all $|\theta|\leqq\pi$, whence $(\lambda I^0-C)^{-1}=A^{-1}\in\mathfrak{K}\subset\mathfrak{B}$, and $(\lambda 1-C)^{-1}=(A_\#)^{-1}=(A^{-1})_\#$ now follows from 3.1. Thus $\lambda\notin\sigma(C_\#)$ when $\lambda\notin\sigma(C)$, and we have proved that $\sigma(C_\#)\subset\sigma(C)$. Lorch [6, Theorem 2] has shown that the spectrum of any member of an irreducible ring is connected. A reference to 4.1 now completes the proof.

4.3. LEMMA. *Suppose $\{F_s\}$ is a family of members of $\mathfrak{B}$. If $A\in\mathfrak{B}$, $A=\lim_{s\to\infty}F_s$, and $T=\lim_{s\to\infty}(F_s)_\#$, then $A_\#=T$.*

PROOF. Take $m, n=0, \pm 1, \pm 2, \cdots$. From (1) follows[3] that

$$[\Phi A]_m = \lim_{s\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}F_s(\theta)d\theta = \lim_{s\to\infty}[\Phi\{F_s\}]_m.$$

Thus, if $x\in S_0$,

$$[(\Phi A)*x]_n = \sum_{\nu=-\infty}^{\infty}[\Phi A]_{n-\nu}x_\nu = \lim_{s\to\infty}[(\Phi\{F_s\})*x]_n.$$

In view of (2), this can be written $[A_\# x]_n=\lim[(F_s)_\# x]_n$. On the other

[3] Note that $A=\lim F_s$ if and only if the sequence $F_s(\theta)$ converges uniformly to $A(\theta)$ on $[-\pi, \pi]$.

hand, $[Tx]_n = \lim\ [(F_s)_\# x]_n$ follows directly from our hypothesis. Combining results, we obtain

$$A_\# x = Tx \qquad \text{for all } x \text{ in } S_0. \tag{4}$$

But $A_\# \in \mathfrak{E}$ and $T \in \mathfrak{E}$ (since the $(F_s)_\#$ are in the complete space $\mathfrak{E}$). The conclusion now follows from (4) and 2.1.

5. **The subring of analyticity.** We say that A is analytic on $[-\pi, \pi]$ if A is holomorphic on an open rectangle $\mathcal{R}(A) \supset [-\pi, \pi]$. Let $\mathfrak{A}$ be the set of all such functions. Note that $\mathfrak{A} \subset \mathfrak{K} \subset \mathfrak{B}$, whence $\{A_\#: A \in \mathfrak{A}\} \subset \mathfrak{K}_\# \subset \mathfrak{B}_\#$; each of these three sets is a subring of $\mathfrak{E}$.

Suppose $A \in \mathfrak{A}$. There exists therefore a sequence $\{F_s\}$ of polynomials such that [9, p. 177 (2.2)]

(a) *the sequence $F_s(\lambda)$ converges uniformly to $A(\lambda)$ on each closed subset of $\mathcal{R}(A)$.*

Let I denote the function $I(\theta) = \theta$; clearly $I \in \mathfrak{K}$ and from 4.2 follows that $\sigma(I_\#) \subset \sigma(I) = [-\pi, \pi]$. Given any B in $\mathfrak{A}$, the Dunford operational calculus associates with the bounded operator $I_\#$ a member $B(I_\#)$ of $\mathfrak{E}$ by means of the equation

$$B(I_\#) = \frac{1}{2\pi i} \int B(\lambda)(\lambda 1 - I_\#)^{-1} d\lambda, \qquad \lambda \in K, \tag{5}$$

where K is the boundary of a rectangle $\mathfrak{D}(A) \supset [-\pi, \pi]$ whose closure is included in $\mathcal{R}(A)$. In case $B(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ is an entire function, then [1, 2.8]

$$B(I_\#) = \sum_{n=0}^{\infty} a_n I_\#^n = \lim_{s \to \infty} \sum_{n=0}^{s} a_n I_\#^n. \tag{6}$$

From (a) follows that[4]

$$A(I_\#) = \lim F_s(I_\#) \qquad (s \to \infty). \tag{7}$$

We shall also need the following result [1, 2.9]

$$A(\sigma(I_\#)) = \{A(\lambda): \lambda \in \sigma(I_\#)\} = \sigma(A(I_\#)) \qquad (A \in \mathfrak{A}). \tag{8}$$

5.1. THEOREM. *If $A \in \mathfrak{A}$, then $A_\# = A(I_\#)$.*

PROOF. Observe that I^n is the function $I^n(\theta) = \theta^n$, so that the polynomial F_s can be written as a finite sum $\sum a_n^{(s)} I^n$ of functions. From 3.2 follows $(F_s)_\# = \sum a_n^{(s)} I_\#^n = F_s(I_\#)$ (for the last equality, use (6)), which enables us to deduce from (7) that $A(I_\#) = \lim\ (F_s)_\#$. On the

[4] See [7, p. 29]; (7) is an immediate consequence of (5). For a more general viewpoint involving (a) and the derivation of (7), see [5, p. 121].

other hand, $A = \lim F_s$ is a consequence[3] of (a), and the conclusion is now obtained from 4.3.

6. **The Titchmarsh operator.** It is easily verified that $I_\#$ maps every member c of l_p on some sequence b such that

$$[I_\# c]_n = b_n = \sum_{\nu=-\infty, \nu\neq n}^{\infty} \frac{i(-1)^{n+\nu}}{n-\nu} c_\nu \qquad (n = 0, \pm 1, \pm 2, \pm 3, \cdots).$$

By 5.1 and (6), the function E^α defined by $E^\alpha(\theta) = e^{-i\alpha\theta}$ satisfies

$$E_\#^\alpha = E^\alpha(I_\#) = \sum_{n=0}^{\infty} a_n I_\#^n; \qquad a_n = (-i\alpha)^n/n!. \tag{9}$$

From 2.2 we have $E_\#^\alpha E_\#^\lambda = (E^\alpha \cdot E^\lambda)_\# = E_\#^{\alpha+\lambda}$ for complex α and λ; this was first proved by Titchmarsh in a special case.[5] Suppose $\alpha = 0, \pm 1, \pm 2, \pm 3, \cdots$; it is readily verified that $E_\#^\alpha$ maps any $c = \{c_n\}$ of l_p on the sequence $\{b_n\}$ such that

$$[E_\#^\alpha c]_n = b_n = c_{n+\alpha} \qquad (n = 0, \pm 1, \pm 2, \pm 3, \cdots). \tag{10}$$

6.1. Notation. When $T \in \mathfrak{E}$, then λ is in the point spectrum $p(T)$ of T if $Tc = \lambda c$ for some c in l_p with $c \neq o$ (where o is the sequence x having $x_n = 0$ for all n).

6.2. Theorem. $p(I_\#) = 0$.

Proof. By way of contradiction, assume $\theta \in p(I_\#)$. Then $I_\# c = \theta c$ for some c in l_p with $c \neq o$, whence $I c_\#^n = \theta^n c$. From (9) now follows $E_\#^\alpha c = \sum_{n=0}^\infty a_n I_\#^n c = (\sum_{n=0}^\infty a_n \theta^n) c$. Therefore $E_\#^\alpha c = E^\alpha(\theta) c$ and $c_m \neq 0$ for some m (since $c \neq o$). Accordingly, $c_{m+\alpha} = E^\alpha(\theta) c_m$ (by (10)), and $|c_{m+\alpha}| = |E^\alpha(\theta)| \, |c_m|$. But $c \in l_p$ and $|E^\alpha(\theta)| = 1$, which implies the absurdity

$$\infty > \|c\|_p^p = \sum_{\alpha=-\infty}^{\infty} |c_{m+\alpha}|^p = \sum_{\alpha=-\infty}^{\infty} |c_m|^p = \infty.$$

6.3. Theorem. *The spectrum $\sigma(E_\#^1)$ is the whole circumference Γ_1 of the unit circle $|\lambda| = 1$.*

Proof. Suppose $\lambda \in \Gamma_1$, and write $T_\lambda = \lambda 1 - E_\#^1$. It will suffice to exhibit a sequence $\{u_s\}$ such that

$$\{u_s\} \subset l_p \quad \text{and} \quad \lim_{s\to\infty} \|u_s\|_p^{-p} \|T_\lambda u_s\|_p^p = 0. \tag{11}$$

[5] See G. L. Krabbe, *The Titchmarsh semi-group*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 219–225.

Indeed, this ensures that $\lambda \in \sigma(E^1_\#)$ (see [5, 2.14.3]); therefore $\Gamma_1 \subset \sigma(E^1_\#) \subset \sigma(E^1) = \Gamma_1$ (the last inclusion follows from 4.2), which concludes the proof.

Set $r_s = \exp(-1/s)$, and define u_s as the sequence b such that $b_n = (\lambda \cdot r_s)^n$ if $n \geqq 0$, while $b_n = 0$ when $n < 0$. When $s > 0$ we thus have $0 < r_s^p < 1$ and

$$\|u_s\|_p^p = \sum_{n=0}^{\infty} (r_s^p)^n = (1 - r_s^p)^{-1} < \infty.$$

On the other hand, $[T_\lambda b]_n$ equals -1 if $n = -1$, and otherwise equals $\lambda b_n - b_{n+1} = b_n \lambda (1 - r_s)$ (see (10)). Consequently,

$$\|u_s\|_p^{-p} \cdot \|T_\lambda u_s\|_p^p = (1 - r_s^p) + |1 - r_s|^p,$$

which approaches zero when $s \to \infty$. Thus (11) is satisfied, and the proof is completed.

6.4. THEOREM.[6] $\sigma(I_\#) = [-\pi, \pi.]$

PROOF. Let $\mathfrak{N} < \mathfrak{M}$ stand for $\mathfrak{N} \subset \mathfrak{M}$, $\mathfrak{N} \neq \mathfrak{M}$. From 4.2 follows that $\sigma(I_\#)$ is a connected subset of $\sigma(I) = [-\pi, \pi]$; being a closed set, $\sigma(I_\#)$ must be a closed interval $\mathcal{I}$ included in $[-\pi, \pi]$. Suppose $\mathcal{I} \neq [-\pi, \pi]$. We then have $\mathcal{I} < [-\pi, \pi]$, whence $\mathcal{I} < \mathcal{I}' < [-\pi, \pi]$ for some interval $\mathcal{I}'$; but E^1 is then a one-to-one mapping of $\mathcal{I}'$ into Γ_1, so that $E^1(\mathcal{I}) < E^1(\mathcal{I}') \subset \Gamma_1$. Recalling that $\mathcal{I} = \sigma(I^1_\#)$, we have $E^1(\mathcal{I}) = \sigma(E^1(I_\#)) = \sigma(E^1_\#)$ by (8) and 5.1, whence $\sigma(E)^1_\# < \Gamma_1$, which contradicts 6.3 and thus disproves our assumption $\mathcal{I} \neq [-\pi, \pi]$.

7. **Conclusion.** Suppose $A \in \mathfrak{A}$. From (8) and 5.1 it follows that $A(\sigma(I_\#)) = \sigma(A_\#)$; by 6.4 we have $A([-\pi, \pi]) = \sigma(A_\#)$, and therefore $\sigma(A_\#)$ is the analytic curve $\sigma(A)$ (since $\sigma(A)$ is the image of $[-\pi, \pi]$ by A). Note that

(vi) *if A is real-valued, then $\sigma(A_\#) = [\alpha_1, \alpha_2]$, where α_1 and α_2 are the extrema of $A(\theta)$ when $|\theta| \leqq \pi$.*

This is the form of the corresponding Hilbert space result of Toeplitz, as given by F. Riesz [8, p. 153].

7.1. THEOREM. *If $A \in \mathfrak{A}$, then $p(A_\#)$ has at most one member; if A is not a constant, then $p(A_\#)$ is empty.*

PROOF. Suppose first that A is a constant: $A \in \mathfrak{A}_0 = \{\zeta I^0 : \zeta \text{ complex}\}$. Then $A_\# = \zeta \mathbf{1}$ (see §3) and $p(A_\#) = \{\zeta\}$. Next, assume $A \notin \mathfrak{A}_0$

[6] The author is much indebted to Professor C. B. Morrey, Jr., who first proved this result.

and $\mu \in p(A_\#)$. Accordingly, the function $C(\lambda) = \mu - A(\lambda)$ does not vanish everywhere on the rectangle $\mathfrak{R}(A)$ of §5; but C is analytic on $\mathfrak{R}(A)$ and has therefore the finite collection $\{\alpha_n : n\}$ of zeros in the rectangle $\mathfrak{D}(A)$ described in §5. The function $B(\lambda) = [C(\lambda)]^{-1} \prod_n (\alpha_n - \lambda)$ is analytic[7] on $\mathfrak{D}(A)$, so that $B \in \mathfrak{B}$. In view of 2.2, we can now deduce from $B \cdot C = B \cdot (\mu I^0 - A) = \prod_n (\alpha_n I^0 - I)$ that

$$B_\#(\mu 1 - A_\#)c = \left[\prod_n (\alpha_n 1 - I_\#) \right] c \qquad \text{for all } c \text{ in } l_p.$$

But $(\mu 1 - A_\#)c = o$ for some $c \neq o$ (since $\mu \in p(A_\#)$). Consequently $o = [\prod_n (\alpha_n 1 - I_\#)]c$; this implies $(\alpha_n 1 - I_\#)c = o$ for some n, whence the contradiction $\alpha_n \in p(I_\#)$ of 6.2.

Bibliography

1. N. Dunford, *Spectral theory* I. *Convergence to projections*, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 185–217.

2. G. H. Hardy and W. W. Rogosinski, *Fourier series*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 38, Cambridge, 1950.

3. P. Hartman and A. Wintner, *The spectra of Toeplitz's matrices*, Amer. J. Math. vol. 76 (1954) pp. 867–882.

4. ———, *On the spectra of Toeplitz's matrices*, Amer. J. Math. vol. 72 (1950) pp. 359–366.

5. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications vol. 31, New York, 1948.

6. E. R. Lorch, *The theory of analytic functions in normed abelian vector rings*, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 414–425.

7. R. S. Phillips, *Semi-groups of operators*, Bull. Amer. Math. Soc. vol. 61 (1955) pp. 16–33.

8. F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues*, Paris, 1913.

9. S. Saks and A. Zygmund, *Analytic functions*, Monografie Matematyczne, Warsaw, 1952.

10. S. B. Stečkin, *On bilinear forms*, C. R. (Doklady) Acad. Sci. URSS. N. S. vol. 71 (1950) pp. 237–240.

11. O. Toeplitz, *Zur Theorie der quadratischen Formen von unendlichvielen Veränderlichen*, Nachr. Ges. Wiss. Göttingen (1910) pp. 489–506.

12. ———, *Zur Theorie der quadratischen und bilinearen Formen von unendlichvielen Veränderlichen*, Math. Ann. vol. 70 (1911) pp. 351–376.

13. ———, *Zur Transformation der Scharen bilinearer Formen von unendlichvielen Veränderlichen*, Nachr. Ges. Wiss. Göttingen (1907) pp. 110–115.

Purdue University

---

[7] Each factor $(\alpha_n - \lambda)$ is repeated m_n times (m_n is the multiplicity of the point α_n). Note that there exists some α_n, since $\mu \in p(A_\#) \subset \sigma(A(I_\#)) = A(\sigma(I_\#))$, which implies $\mu - A(\alpha_n) = 0$ for some α_n in $\sigma(I_\#) \subset \mathfrak{D}(A)$.