PARTITIONS OF MULTI-PARTITE NUMBERS

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1. Introduction. In what follows all small latin letters denote non-negative rational integers. We suppose for the present that $|X_i| < 1$ $(1 \leqq i \leqq j)$ and write

$$F_j(Y) = F_j(X_1, \cdots, X_j; Y) = \prod (1 + X_1^{k_1} \cdots X_j^{k_j} Y)$$

and

$$G_j(Y) = \{F_j(-Y)\}^{-1} = \prod (1 - X_1^{k_1} \cdots X_j^{k_j} Y)^{-1},$$

where the products extend over all non-negative $k_1, \cdots, k_j$. If $|Y| < 1$, we have

$$G_j(Y) = 1 + \sum_{n=1}^{\infty} Q_j(n) Y^n,$$

where

$$Q_j(n) = Q_j(X_1, \cdots, X_j; n) = \sum_{n_1, \cdots, n_j = 0}^{\infty} q(n_1, \cdots, n_j; n) X_1^{n_1} \cdots X_j^{n_j}$$

and $q(n_1, \cdots, n_j; n)$ is the number of partitions of the j-partite number $(n_1, \cdots, n_j)$ into just n parts, that is the number of solutions of the "vector" equation (or equation in single row matrices)

$$\sum_{k=1}^{n} (x_{1k}, \cdots, x_{jk}) = (n_1, \cdots, n_j). \tag{1}$$

The order of the vectors on the left-hand side of (1) is irrelevant. Again

$$F_j(Y) = 1 + \sum_{n=1}^{\infty} R_j(n) Y^n,$$

where

$$R_j(n) = \sum r(n_1, \cdots, n_j; n) X_1^{n_1} \cdots X_j^{n_j}$$

and $r(n_1, \cdots, n_j; n)$ is the number of partitions of $(n_1, \cdots, n_j)$ into just n *different* parts, that is, the number of solutions of (1) in which the vectors on the left hand side are all different.

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If $j=1$, we have

$$(1 - Y)G_1(Y) = G_1(X_1Y)$$

and so

$$Q_1(n) - Q_1(n-1) = X_1^n Q_1(n),$$

whence

$$Q_1(n) = \frac{Q_1(n-1)}{1 - X_1^n} = \frac{1}{(1 - X_1)(1 - X_1^2) \cdots (1 - X_1^n)}.$$

Similarly we find that

$$R_1(n) = \frac{X_1^{n(n-1)/2}}{(1 - X_1)(1 - X_1^2) \cdots (1 - X_1^n)}.$$

Macmahon (*Combinatory analysis* ii, Cambridge, 1916) discussed in detail the case $j=1$ and referred briefly to the more general case, commenting on its complexity. More recently Bellman (Bull. Amer. Math. Soc. Research Problem 61-1-3) has asked for a formula for $Q_2(n)$. My object here is to obtain formulae for $Q_j(n)$ and $R_j(n)$ for general j and n. For $j>1$, these formulae cannot be reduced to anything as simple as in the case $j=1$, but we can make some progress in this direction and deduce certain results about partitions.

2. **The formulae for $Q_j(n)$ and $R_j(n)$.** Let

$$\alpha_1, \alpha_2, \alpha_3, \cdots \tag{2}$$

be any infinite sequence such that $|\alpha_k| < 1$ for every k and $\sum |\alpha_k| < \infty$. We write

$$C(Y) = \prod_{k=1}^{\infty} (1 + \alpha_k Y) = 1 + \sum_{n=1}^{\infty} A(n) Y^n$$

and

$$D(Y) = \{C(-Y)\}^{-1} = \prod_{k=1}^{\infty} (1 + \alpha_k Y + \alpha_k^2 Y^2 + \cdots)$$
$$= 1 + \sum_{n=1}^{\infty} B(n) Y^n.$$

Clearly $A(n)$ is the sum of the products of every set of n *different* α and $B(n)$ is the sum of the products of every set of n numbers α, repetitions permitted. We write also

$$S(m) = \sum_k \alpha_k^m.$$

We see at once that

$$\log D(Y) = -\sum_{k=1}^{\infty} \log (1 - \alpha_k Y) = \sum_{m=1}^{\infty} \frac{S(m)}{m} Y^m. \tag{3}$$

Hence

$$D(Y) = \exp \left\{ \sum_{m=1}^{\infty} \frac{S(m)}{m} Y^m \right\}$$

and

$$B(n) = \sum_{(n)} \prod \frac{\{S(m)\}^{h_m}}{h_m! m^{h_m}},$$

the sum extending over all partitions of n of the form

$$n = \sum h_m m$$

and the product over all the different parts m in the partition. Again

$$C(-Y) = \exp \left\{ -\sum_{m=1}^{\infty} \frac{S(m)}{m} Y^m \right\}$$

and so

$$A(n) = (-1)^n \sum_{(n)} \prod \frac{(-1)^{h_m} \{S(m)\}^{h_m}}{h_m! m^{h_m}}.$$

Next, if we differentiate (3) with respect to Y and multiply through by $D(Y)$, we obtain

$$\sum_{n=1}^{\infty} nB(n) Y^{n-1} = \sum_{m=1}^{\infty} S(m) Y^{m-1} \left\{ 1 + \sum_{n=1}^{\infty} B(n) Y^n \right\}$$

and so, equating coefficients of Y^{n-1}, we have

$$nB(n) = \sum_{m=1}^{n} S(m) B(n-m). \tag{4}$$

Similarly

$$nA(n) = \sum_{m=1}^{n} (-1)^{m-1} S(m) A(n-m). \tag{5}$$

If we now take all of

$$X_1^{k_1} \cdots X_j^{k_j} \qquad (k_i \geqq 0,\ 1 \leqq i \leqq j)$$

for the α in (2), we see that

$$A(n) = R_j(n), \qquad B(n) = Q_j(n)$$

and

$$S(m) = \sum X_1^{mk_1} \cdots X_j^{mk_j} = \prod_{i=1}^{j} \left(\frac{1}{1 - X_i^m} \right) = \frac{1}{\beta_j(m)},$$

where

$$\beta_j(m) = \prod_{i=1}^{j} (1 - X_i^m).$$

Hence we have

$$Q_j(n) = \sum_{(n)} \prod (h_m!)^{-1} \{ m\beta_j(m) \}^{-h_m} \tag{6}$$

and

$$R_j(n) = (-1)^n \sum_{(n)} \prod (-1)^{h_m} (h_m!)^{-1} \{ m\beta_j(m) \}^{-h_m}. \tag{7}$$

These are the formulae for $Q_j(n)$ and $R_j(n)$. For $j=1$, they were found by Macmahon (loc. cit.).

Again, (4) and (5) become

$$nQ_j(n) = \sum_{m=1}^{n} \frac{Q_j(n-m)}{\beta_j(m)} \tag{8}$$

and

$$nR_j(n) = \sum_{m=1}^{n} (-1)^n \frac{R_j(n-m)}{\beta_j(m)}$$

If $\sum h_m\, m = n$, it is easy to show that

$$\frac{(1 - X)(1 - X^2) \cdots (1 - X^n)}{\prod (1 - X^m)^{h_m}}$$

is a polynomial in X. Its degree is clearly

$$\sum_{h=1}^{n} k - \sum h_m m = \frac{1}{2} n(n+1) - n = \frac{1}{2} n(n-1).$$

Hence, if we write

$$P_j(n) = P_j(X_1, \cdots, X_j; n) \\ = \beta_j(1)\beta_j(2) \cdots \beta_j(n)Q_j(X_1, \cdots, X_j; n) \tag{9}$$

we see from (6) that $P_j(n)$ is a polynomial of degree at most $n(n-1)/2$ in each of $X_1, \cdots, X_j$.

It follows from its definition that $Q_j(n)$ is a multiple infinite power series in $X_1, \cdots, X_j$, the coefficient of each term being a non-negative integer. Since the β are polynomials with integral coefficients, we see that all the coefficients in the polynomial $P_j(n)$ are integers. It seems very likely that all these coefficients are non-negative, but this I have not been able to prove. In §1, we saw that

$$P_1(n) = 1 \tag{10}$$

for all n. Unfortunately nothing so simple is true for $j>1$.

3. Properties of $P_j(n)$. We now suppose that $Q_j(n)$ and $R_j(n)$ are *defined* by (6) and (7), so that $Q_j(n)$ and $R_j(n)$ are rational functions defined for all values of the X_i except the mth roots of unity for which $1 \leqq m \leqq n$. Again, since $P_j(n)$ is a polynomial, it can be defined for all values of the X_i without exception. We write $P_j(0) = Q_j(0) = R_j(0) = 1$ and see that $P_j(1) = 1$.

We have now

$$\beta_j(X_1, \cdots, X_{j-1}, X_j^{-1}; m) = (1 - X_1^m) \cdots (1 - X_{j-1}^m)(1 - X_j^{-m}) \\ = - X_j^{-m}\beta_j(X_1, \cdots, X_{j-1}, X_j; m).$$

Hence, by (6) and (7),

$$Q_j(X_1, \cdots, X_{j-1}, X_j^{-1}; n) = X_j^n \sum_{(n)} \prod (-1)^{h_m}(h_m!)^{-1}\{m\beta_j(m)\}^{-h_m} \\ = (-1)^n X_j^n R_j(X_1, \cdots, X_{j-1}, X_j; n)$$

and

$$R_j(X_1, \cdots, X_{j-1}, X_j^{-1}; n) = (-1)^n X_j^n Q_j(X_1, \cdots, X_j; n).$$

This transformation applies also with any one of $X_1, \cdots, X_{j-1}$ in place of X_j. Applying it twice, we have

$$Q_j(X_1, \cdots, X_{j-2}, X_{j-1}^{-1}, X_j^{-1}; n) = X_{j-1}^n X_j^n Q_j(X_1, \cdots, X_j; n). \tag{11}$$

Using (9), we see that

$$R_j(X_1, \cdots, X_j; n) = \frac{X_j^{n(n-1)/2} P_j(X_1, \cdots, X_{j-1}, X_j^{-1}; n)}{\beta_j(1) \cdots \beta_j(n)},$$

so that, if we can evaluate $P_j(n)$, we have a simple form for both $Q_j(n)$ and $R_j(n)$. Again

$$(X_{j-1}X_j)^{n(n-1)/2}P_j(X_1, \cdots, X_{j-2}, X_{j-1}^{-1}, X_j^{-1}; n) = P_j(X_1, \cdots, X_j; n).$$

If, then, we write

$$g = n(n-1)/2$$

and

$$P_j(X_1, \cdots, X_j; n) = \sum_{k_1, \cdots, k_j = 0}^{g} \lambda(k_1, \cdots, k_j) X_1^{k_1} \cdots X_j^{k_j},$$

we have

$$\lambda(k_1, \cdots, k_{j-2}, k_{j-1}, k_j) = \lambda(k_1, \cdots, k_{j-2}, g - k_{j-1}, g - k_j)$$

and similarly for any other pair of the k_i. We can see at once by putting $X_1 = X_2 = \cdots = X_j = 0$, that $\lambda(0, 0, \cdots, 0) = 1$. Hence $\lambda(g, g, 0, 0, \cdots, 0) = 1$ and so on. It follows that, for $j \geqq 2$, $P_j(n)$ is of degree exactly g in every X_i.

Next, we see that, in the sum on the right-hand side of (6), the factor $(1-X_j)^n$ occurs in the denominator only in the term in which $m=1$, $h_1 = n$, i.e. the term corresponding to the partition of n into n units. But the factor $(1-X_j)^n$ occurs in $\beta_j(1)\beta_j(2) \cdots \beta_j(n)$ and so

$$\begin{aligned} P_j(X_1, \cdots, X_{j-1}, 1; n) &= \lim_{X_j \to 1} \frac{\beta_j(1) \cdots \beta_j(n)}{n!\{\beta_j(1)\}^n} \\ &= \frac{\beta_{j-1}(1) \cdots \beta_{j-1}(n)}{\{\beta_{j-1}(1)\}^n} \qquad (12) \\ &= \prod_{i=1}^{j-1} \prod_{m=2}^{n} (1 + X_i + X_i^2 + \cdots + X_i^{m-1}). \end{aligned}$$

Hence

$$\sum_{k_j=0}^{g} \lambda(k_1, \cdots, k_j)$$

is the coefficient of $\prod_{i=1}^{j-1} X_i^{k_i}$ in the double product on the right-hand side of (12). Also, putting

$$X_1 = X_2 = \cdots = X_{j-1} = 1,$$

we have

$$P_j(1, 1, \cdots, 1; n) = \sum_{k_1, \cdots, k_j = 0}^{g} \lambda(k_1, \cdots, k_j) = (n!)^{j-1}.$$

Again

$$P_j(X_1, \cdots, X_{j-1}, 0; n) = P_{j-1}(X_1, \cdots, X_{j-1}; n)$$

and so

$$\lambda(k_1, \cdots, k_{j-1}, 0) = \lambda(k_1, \cdots, k_{j-1}).$$

By (10), $\lambda(k_1)=0$ unless $k_1=0$. Hence

$$\lambda(k_1, 0, 0, \cdots, 0) = 0,$$

unless $k_1=0$. Thus there is no term in P_j which consists of a power of one X only, i.e. apart from the term of zero degree, viz. 1, every term contains at least two of the X. A number of other properties of the λ may be obtained similarly.

From (8) and (9), it follows that

$$nP_j(n) = \sum_{m=1}^{n} \frac{\beta_j(n-m+1)\cdots\beta_j(n)}{\beta_j(m)} P_j(n-m). \tag{13}$$

For $m \geqq 2$, the factor $1-X_i$ occurs at least once more in the numerator of

$$\frac{\beta_j(n-m+1)\cdots\beta_j(n)}{\beta_j(m)}$$

than in the denominator. Hence

$$nP_j(n) = \frac{\beta_j(n)}{\beta_j(1)} P_j(n-1) + \beta_j(1)T,$$

where T is a polynomial in the X_i.

For a small value of m, we can find the terms containing X_j^m in $P_j(n)$ as follows. It is easily verified that

$$G_j(X_jY)G_{j-1}(Y) = G_j(Y)$$

and so

$$\sum_{n=0}^{\infty} Q_j(n)Y^n = \left\{\sum_{l=0}^{\infty} Q_j(l)X_j^lY^l\right\}\left\{\sum_{s=0}^{\infty} Q_{j-1}(s)Y^s\right\},$$

whence

$$(1-X_j^n)Q_j(n) = \sum_{l=0}^{n-1} X_j^l Q_j(l)Q_{j-1}(n-l),$$

that is

$$(14)\qquad P_j(n) = \sum_{l=0}^{n-1} X_j^l \left\{ \prod_{m=l+1}^{n-1} (1 - X_j^m) \right\} \cdot \frac{\beta_{j-1}(n-l+1)\cdots\beta_{j-1}(n)}{\beta_{j-1}(1)\cdots\beta_{j-1}(l)} P_j(l) P_{j-1}(n-l),$$

where, as usual, each empty product denotes unity. The terms in X_j^m occur in the first $m+1$ terms on the right and can be expressed in terms of $P_j(l)$ and $P_{j-1}(n-l)$ for $1 \leqq l \leqq m$. Thus the term in X_j is

$$X_j \left\{ \frac{\beta_{j-1}(n)}{\beta_{j-1}(1)} P_{j-1}(n-1) - P_{j-1}(n) \right\}.$$

4. **Calculation of $P_j(2)$ and $P_j(3)$.** By (6) and (9),

$$\begin{aligned} P_j(2) &= \frac{\beta_j(1)\beta_j(2)}{2} \left\{ \frac{1}{\{\beta_j(1)\}^2} + \frac{1}{\beta_j(2)} \right\} \\ &= \frac{1}{2}\left(\frac{\beta_j(2)}{\beta_j(1)} + \beta_j(1) \right) \\ &= \frac{1}{2}\left\{ \prod_{i=1}^{j} (1 + X_i) + \prod_{i=1}^{j} (1 - X_i) \right\} \\ &= 1 + \sum X_1X_2 + \sum X_1X_2X_3X_4 + \cdots. \end{aligned}$$

Similarly, since $3 = 2+1 = 1+1+1$, we have

$$\begin{aligned} P_j(3) &= \beta_j(1)\beta_j(2)\beta_j(3) \left\{ \frac{1}{6\{\beta_j(1)\}^3} + \frac{1}{2\beta_j(1)\beta_j(2)} + \frac{1}{3\beta_j(3)} \right\} \\ &= \frac{1}{6}\left\{ \frac{\beta_j(2)\beta_j(3)}{\{\beta_j(1)\}^2} + 3\beta_j(3) + 2\beta_j(1)\beta_j(2) \right\} \\ &= \frac{1}{6}\left\{ \prod (1 + X_i)(1 + X_i + X_i^2) + 3\prod (1 - X_i^3) \right. \\ &\qquad \left. + 2\prod (1 - X_i)(1 - X_i^2) \right\}. \end{aligned}$$

Now

$$\begin{aligned} &\prod (1 + 2X_i + 2X_i^2 + X_i^3) + 3\prod (1 - X_i^3) + 2\prod (1 - X_i - X_i^2 + X_i^3) \\ &\quad = 3\prod (1 + X_i^3) + 3\prod (1 - X_i^3) + \sum_{a=2}^{j} \{2^a + (-1)^a 2\} \\ &\qquad \cdot \sum X_1 \cdots X_a (1 + X_1) \cdots (1 + X_a)(1 + X_{a+1}^3) \cdots (1 + X_j^3) \end{aligned}$$

and so

$$P_j(3) = 1 + \sum X_1^3 X_2^3 + \sum X_1^3 X_2^3 X_3^3 X_4^3 + \cdots$$
$$+ X_1 \cdots X_j(1 + X_1) \cdots (1 + X_j) \sum_{b=0}^{j-2} \frac{1}{3} \{2^{j-b-1} + (-1)^{j-b}\}$$
$$\cdot \sum \left(X_1 - 1 + \frac{1}{X_1}\right) \cdots \left(X_b - 1 + \frac{1}{X_b}\right).$$

Macmahon (loc. cit.) gives the above form of $P_j(2)$, but dismisses $P_j(3)$ with the remark that it is very complex.

From the above, we have

$$P_2(2) = 1 + X_1X_2, \qquad P_3(2) = 1 + X_1X_2 + X_2X_3 + X_3X_1$$

and

$$P_2(3) = 1 + X_1^3X_2^3 + X_1X_2(1 + X_1)(1 + X_2),$$
$$P_3(3) = 1 + X_1^3X_2^3 + X_2^3X_3^3 + X_3^3X_1^3$$
$$+ X_1X_2X_3(1 + X_1)(1 + X_2)(1 + X_3)\left\{\sum X_1 - 2 + \sum \frac{1}{X_1}\right\}.$$

The formulae (6) and (9) enable one to evaluate $P_j(n)$ for small j and n and, in particular, to pick out the coefficient of any given term.

5. **The case** $j=2$. By (12), we see that

$$P_2(X_1, 1; n) = \prod_{m=2}^{n} (1 + X_1 + X_1^2 + \cdots + X_1^{m-1}) = \prod_{m=2}^{n} \left(\frac{1 - X_1^m}{1 - X_1}\right).$$

We see then that

$$P_2(X_1, X_2; n) - \prod_{m=2}^{n} \left(\frac{1 - X_1^m X_2^m}{1 - X_1X_2}\right)$$

vanishes when $X_2 = 1$ and similarly when $X_1 = 1$. It also vanishes in virtue of (10), when $X_1 = 0$ and when $X_2 = 0$. It follows that

$$P_2(X_1, X_2; n) = \prod_{m=2}^{n} \left(\frac{1 - X_1^m X_2^m}{1 - X_1X_2}\right) \tag{15}$$
$$+ X_1X_2(1 - X_1)(1 - X_2)M(X_1, X_2; n),$$

where M is a polynomial in X_1 and X_2. Since

$$X_1^g X_2^g P_2(X_1^{-1}, X_2^{-1}; n) = P_2(X_1, X_2; n)$$

and a similar relation is true for the first term on the right-hand side of (15), we must have

$$X_1^{g-3}X_2^{g-3}M(X_1^{-1}, X_2^{-1}; n) = M(X_1, X_2; n)$$

and so M is of degree at most $g-3$ in X_1 and X_2.

For a fixed j, the recurrence formula (13) provides a slightly less laborious means of finding $P_j(n)$ than does (6). If we write $Z=X_1X_2$ and

$$\zeta_m = 1 + Z + Z^2 + \cdots + Z^m,$$

the values of $P_2(4)$ and $P_2(5)$ found from (13) are

$$\begin{aligned} P_2(4) &= \zeta_1\zeta_2\zeta_3 - Z\zeta_3\beta_2(1) - Z\zeta_2\beta_2(2) \\ &= (1 + Z^2)\zeta_4 + Z\zeta_3(X_1 + X_2) + Z\zeta_2(X_1^2 + X_2^2) \end{aligned}$$

and

$$\begin{aligned} P_2(5) &= \zeta_1\zeta_2\zeta_3\zeta_4 - Z\zeta_3\zeta_4\beta_2(1) - Z\zeta_2\zeta_4\beta_2(2) \\ &\quad - Z(1 + Z^2)\zeta_3\beta_2(3) - Z^2\zeta_2\beta_2(4) \\ &= 1 + Z + 2Z^2 + 3Z^3 + 4Z^4 + 6Z^5 + 4Z^6 \\ &\quad + 3Z^7 + 2Z^8 + Z^9 + Z^{10} \\ &\quad + Z\zeta_3\zeta_4(X_1 + X_2) + Z\zeta_2\zeta_4(X_1^2 + X_2^2) \\ &\quad + Z(1 + Z^2)\zeta_3(X_1^3 + X_2^3) + Z^2\zeta_2(X_1^4 + X_2^4). \end{aligned}$$

The detailed calculations have no point of interest.

6. Consequences in partition-theory. If

$$\frac{1}{(1 - X)(1 - X^2) \cdots (1 - X^n)} = 1 + \sum_{t=1}^{\infty} p_n(t)X^t,$$

then $p_n(t)$ is the number of partitions of t into parts not greater than n. It is well known (see, for example, Hardy and Wright, *Theory of numbers*, 3d ed., Oxford, 1955, Theorem 343) that $p_n(t)$ is also the number of partitions of t into not more than n parts. From the definition of $Q_j(n)$ and $P_j(n)$, we see that

$$q(n_1, \cdots, n_j; n) = \sum_{k_1, \cdots, k_j=0}^{g} \lambda(k_1, \cdots, k_j) \prod_{i=1}^{j} p_n(n_i - k_i).$$

Hence, if we calculate $P_j(n)$, we can express $q(n_1, \cdots, n_j; n)$ in terms of the p_n. Again

$$r(n_1, \cdots, n_j; n) = \sum_{k_1, \cdots, k_j=0}^{g} \lambda(k_1, k_2, \cdots, k_{j-1}, g - k_j) \prod_{i=1}^{j} p_n(n_i - k_i).$$

7. An asymptotic expansion for large n. For fixed X_i such that $|X_i|<1$ $(1\leqq i\leqq j)$ we can find an asymptotic expansion of $Q_j(n)$ for large n. For simplicity, we confine ourselves to the case in which $j=2$, X_1 and X_2 are real and positive and the ratio of their logarithms is not rational, so that $X_1^u=X_2^v$ is impossible for any positive integral u and v. In the complex Y-plane, $G_2(Y)$ has a simple pole at each of the points

$$X_1^{-t_1}X_2^{-t_2} \qquad (t_1,\, t_2 \geqq 0).$$

If we write $\delta=\min(|X_1|^{-1}, |X_2|^{-1})$,

$$\phi(\alpha, X) = \prod_{k=0}^{\infty}(1-\alpha X^k)^{-1},$$

$$J = \phi(X_1, X_1)\phi(X_2, X_2)\prod_{k_1=1}^{\infty}\prod_{k_2=1}^{\infty}\left(1-X_1^{k_1}X_2^{k_2}\right)^{-1},$$

and

$$K(t_1, t_2; X_1, X_2)$$

$$=\prod_{k_1=1}^{t_1}\prod_{k_2=1}^{t_2}\left(1-X_1^{-k_1}X_2^{-k_2}\right)^{-1}\prod_{k_2=1}^{t_2}\phi(X_2^{-k_2}, X_1)\prod_{k_1=1}^{t_1}\phi(X_1^{-k_1}, X_2),$$

we find that

$$G_2(Y) - J\sum_{h=0}^{m+1}\sum_{k_1=0}^{h}\frac{K(k_1,\, h-k_1; X_1, X_2)}{1-X_1^{k_1}X_2^{h-k_1}Y}$$

is regular on and within the circle $|Y|=\delta^{m+1}$. It follows that

$$Q_2(n) = J\sum_{h=0}^{m}\sum_{k_1=0}^{h}K(k_1,\, h-k_1; X_1, X_2)X_1^{nk_1}X_2^{n(h-k_1)} + O(\delta^{n(m+1)}),$$

where the $O(\ )$ symbol refers to the passage of n to infinity.

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