ON A GENERALIZATION OF THE POISSON FORMULA FOR THE SOLUTION OF THE HEAT FLOW EQUATION

ARNOLD J. TINGLEY

1. **Introduction.** In this note we consider the following boundary value problem;

$$(1a) \quad \Delta G - a \frac{\partial G}{\partial t} + \theta(t, \bar{\xi})G = 0,$$

$$(1b) \quad \lim_{t \rightarrow 0^+} G(t, \xi) = \sigma(\bar{\xi}) \text{ for almost all } \xi.$$

Here $\bar{\xi}$ denotes the variable $(\xi_1, \xi_2, \dots, \xi_n)$ which ranges over the n -dimensional Euclidean space E^n and has the norm $|\bar{\xi}| = (\xi_1^2 + \xi_2^2 + \dots + \xi_n^2)^{1/2}$, the number t has the range $0 < t < t_0$ where t_0 is a positive number or $+\infty$, a is a positive constant, and the symbol Δ denotes the n -dimensional Laplace operator.

Under suitable assumptions on $\theta(t, \bar{\xi})$ and $\sigma(\bar{\xi})$, a solution of the system (1a, b) is given in the region $R \equiv \{(t, \bar{\xi}): 0 < t < t_0, \bar{\xi} \in E^n\}$, by

$$(2) \quad G(t, \xi) = I_{\theta, \sigma}(t, \xi),$$

where

$$I_{\theta, \sigma}(t, \bar{\xi}) = \int_{C^n} \exp \left\{ \frac{t}{a} \int_0^1 \theta \left[t(1-s), 2 \left(\frac{t}{a} \right)^{1/2} x_1(s) + \xi_1, \dots, \right. \right. \\ \left. \left. 2 \left(\frac{t}{a} \right)^{1/2} x_n(s) + \xi_n \right] ds \right\} \\ \cdot \sigma \left[2 \left(\frac{t}{a} \right)^{1/2} x_1(1) + \xi_1, \dots, 2 \left(\frac{t}{a} \right)^{1/2} x_n(1) + \xi_n \right] d_w \bar{x},$$

$x_1(s), \dots, x_n(s)$ being elements of the Wiener space C , and the integral being the n -fold Wiener integral over the product space C^n [6].

Alternative assumptions on $\theta(t, \bar{\xi})$ and $\sigma(\bar{\xi})$ under which (1b) may be replaced by the two conditions

$$(3) \quad \lim_{|\bar{\xi}| \rightarrow \infty} G(t, \xi) = 0,$$

$$(4) \quad \lim_{t \rightarrow 0^+} \int_{E^n} |G(t, \xi) - \sigma(\bar{\xi})| d\xi = 0,$$

are also given.

Received by the editors July 11, 1955 and, in revised form, October 24, 1955.

This boundary value problem for the case $n = 1$ has been studied by R. H. Cameron [1]. In this note it is indicated how much of Cameron's work can be extended to n -dimensions while at the same time certain of his conditions can be weakened.

The differential equation (1a) with different boundary conditions has been studied by M. Rosenblatt [5]. His results, as well as his methods, are, however, very different from those considered here.

2. Statement of results. The following theorems sharpen and extend the results given by Cameron in Theorems 2, 3 and 4 of [1].

THEOREM 1 (EXISTENCE THEOREM). *Let $\theta(t, \xi)$ and its partial derivatives $\theta_i(t, \xi)$, $\theta_{\xi_i}(t, \xi)$, ($i = 1, 2, \dots, n$), and $\theta_{\xi_i \xi_i}(t, \xi)$, ($i = 2, 3, \dots, n$) be continuous functions of (t, ξ) in R and satisfy in R the following conditions;*

$$(5) \quad \theta(t, \xi) < \sum_{j=1}^n C_j \xi_j^2 + B_1,$$

$$(6) \quad |\theta_i(t, \xi)| < B_2 \exp \left\{ \sum_{j=1}^n B_{3,j} \xi_j^2 \right\},$$

$$(7) \quad |\theta_{\xi_i}(t, \xi)| < B_2 \exp \left\{ \sum_{j=1}^n B_{3,j} \xi_j^2 \right\}, \quad (i = 1, \dots, n),$$

$$(8) \quad |\theta_{\xi_i \xi_i}(t, \xi)| < B_2 \exp \left\{ \sum_{j=1}^n B_{3,j} \xi_j^2 \right\}, \quad (i = 2, \dots, n),$$

where $B_1, B_2, B_{3,j}$ and C_j are non-negative constants, and where, for each integer j , $1 \leq j \leq n$, $B_{3,j}$ and C_j satisfy the inequality

$$(9) \quad C_j + \frac{a\pi^2}{t_0} B_j < \frac{a^2\pi^2}{16t_0^2}.$$

Assume also that $\sigma(\xi) \exp \left\{ - \sum_{j=1}^n b_j \xi_j^2 \right\}$ is of class L_1 on E^n , where each b_j is a constant satisfying

$$(10) \quad b_j + 4B_{3,j} < ((C_j)^{1/2}/2) \cot \frac{2t_0}{a} (C_j)^{1/2}.$$

Then the function $G(t, \xi)$ defined by (2) satisfies the equation (1a) in R . It also satisfies at each continuity point ξ_0 of $\sigma(\xi)$

$$\lim_{t \rightarrow 0^+, |\xi - \xi_0| \rightarrow 0} G(t, \xi) = \sigma(\xi_0),$$

and, if $\sigma(\xi)$ is bounded on every finite n -dimensional interval, the bound-

any condition (1b) is satisfied for almost all ξ . If no boundedness is required for $\sigma(\xi)$, but (10) is replaced by the stronger inequality

$$b_i \leq (- (C_i)^{1/2}/2) \tan \frac{2t_0}{a}(C_i)^{1/2}$$

then (1b) may be replaced by the two conditions (3) and (4).

THEOREM 2 (UNIQUENESS THEOREM FOR POSITIVE SOLUTIONS FOR $n=1$). Let $\theta(t, \xi)$ be continuous in R , be bounded below in the neighborhood of each point of the edge $t=0$, and satisfy

$$\theta(t, \xi) < C_1 \xi^2 + B_1,$$

where B_1 and C_1 are constants and C_1 is such that $4C_1 t_0^2/a^2 < \pi^2/4$. Let $G(t, \xi)$ be an almost everywhere positive solution of the system (1a, b) in R with continuous derivatives G_t and G_ξ , and corresponding to some constant B satisfying $B < ((C_1)^{1/2}/2) \cot 2t_0(C_1)^{1/2}/a$ let there exist a positive number $t_1 < t_0$ such that

$$\sup_{0 < t < t_1} |G(t, \xi)| \exp \{ -B\xi^2 \}$$

is of class L_1 on E^1 . Then if in each closed substrip of R the ratio

$$\frac{G}{(1+G)(1+|\xi|) \log(2+|\xi|)}$$

is bounded, it follows that G is given uniquely by (2).

THEOREM 3 (GENERAL UNIQUENESS THEOREM FOR $n=1$). In the region R let $\theta(t, \xi)$ be bounded below and satisfy (5), let $\theta_t(t, \xi)$ and $\theta_\xi(t, \xi)$ be continuous and satisfy (6) and (7), and let (9) hold. For each t'_0 satisfying $0 < t'_0 < t_0$ let R' ($R' = R'(t'_0)$) be defined by $R' \equiv \{(t, \xi): 0 < t < t'_0, \xi \in E^1\}$. Then if $G(t, \xi)$ is any solution of the system (1a, b) in R with continuous derivatives G_t and G_ξ such that in each R'

$$|G(t, \xi)| < B_4 \exp \{ B_5 \xi^2 \},$$

where B_4 and B_5 are positive numbers (depending on R' , and thus on t'_0), it follows that G is unique in R . If, moreover, $\sigma(\xi)$ is such that $\sigma(\xi) \exp \{ -b_1 \xi^2 \}$ is measurable and bounded for some b_1 satisfying (10), then such a solution $G(t, \xi)$ does indeed exist and is given by (2).

Conversely, if the system (1a, b) can be solved by other means, the Wiener integral defined by (2) can be calculated. From this point of view we have the following theorem, which is a sharpening of Theorem 5 of [1] and which follows very readily from Theorem 3.

THEOREM 4. Let $\psi(s, u)$ have continuous derivatives ψ_s, ψ_u , in the

region $R^* \equiv \{(s, u): -\delta < s < 1, -\infty < u < \infty\}$ (where δ is positive), and in R^* let ψ, ψ_s, ψ_u , satisfy

$$\begin{aligned}\psi(s, u) &< C_1 u^2/4 + B_1, \\ |\psi_s(s, u)| &< B_2 \exp \{B_{3,1} u^2\}, \\ |\psi_u(s, u)| &< B_2 \exp \{B_{3,1} u^2\},\end{aligned}$$

where $B_{3,1}$ and C_1 satisfy (9). Let $\sigma(u) \exp \{-b_1 u^2\}$ be bounded and measurable on E^1 for some b_1 satisfying

$$b_1 + 4\beta_{3,1} < \frac{1}{2} (C_1)^{1/2} \cdot \cot \frac{\lambda}{2} (C_1)^{1/2} \text{ where } \lambda > 1.$$

Then the following formula holds;

$$\int_c \exp \left\{ \int_0^1 \psi(s, x(s)) ds \right\} \sigma(x(1)) d_w x = H(0, 0),$$

where $H(s, u)$ is the (unique) solution in R^* of the system

$$\frac{1}{4} \frac{\partial^2 H}{\partial u^2} + \frac{\partial H}{\partial s} + \psi(s, u) H = 0,$$

$$\lim_{s \rightarrow 1^-} H(s, u) = \sigma(u) \text{ for almost all real } u.$$

Unfortunately it has not yet been found possible to extend Cameron's uniqueness results to n -dimensions. This is because the transformation theorem due to Cameron and Fagen [2] which is the basis for the proof of the uniqueness theorem only applies to one dimension. It seems probable that future research will overcome this restriction.

3. Methods of proof. The theorems given above can be proved by modifications and extensions of the methods used by Cameron. The main changes are of course due to the weakening of his conditions on $\theta(t, \xi)$ and its derivatives to our conditions (5), (6), (7) and (8), and the extension of part of his work to n -dimensions.

The conditions (5), (6), (7) and (8) lead to factors of the form $\exp \{\lambda \int_0^1 [x(t)]^2 dt\}$ and $\exp \{\mu [x(s)]^2\}$ (where λ and μ are certain constants) in the integrands of certain Wiener integrals. Because of the occurrence of the former factor, it becomes necessary to obtain a formula for evaluating integrals of the form

$$(11) \quad \int_c \exp \left\{ \lambda \int_0^1 [x(s)]^2 ds \right\} F \left(\int_0^1 x(s) ds, x(1) \right) d_w x,$$

where $\lambda < \pi^2/4$.

To accomplish this, we use the following special case of a formula due to Cameron and Martin [3],

$$(12) \quad \int_C F(x) \exp \left\{ \lambda \int_0^1 [x(t)]^2 dt \right\} d_w x \\ = (\sec \lambda^{1/2})^{1/2} \int_C F \left[f(\cdot) \int_0^{\cdot} dx(s)/f(s) \right] d_w x,$$

where $f(s) = \cos \lambda^{1/2}(s-1)$, and Wiener's formula,

$$(13) \quad \int_C f \left[\int_0^1 \alpha_1(t) dx(t), \dots, \int_0^1 \alpha_n(t) dx(t) \right] d_w x \\ = \pi^{-n/2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(u_1, \dots, u_n) \\ \cdot \exp \{ -u_1^2 - \dots - u_n^2 \} du_1 \dots du_n,$$

where $\{\alpha_i(t)\}$ is a real orthonormal set on $[0, 1]$ such that each $\alpha_i(t)$ is of B.V.

Using (12), (11) becomes

$$(14) \quad (\sec \lambda^{1/2})^{1/2} \int_C F \left[\int_0^1 \left(f(s) \int_0^s dy(r)/f(r) \right) ds, f(1) \int_0^1 dy(r)/f(r) \right] d_w y \\ = (\sec \lambda^{1/2})^{1/2} \int_C F \left[- \int_0^1 (g(r)/f(r)) dy(r), \int_0^1 dy(r)/f(r) \right] d_w y,$$

where $g(r) = (\sin \lambda^{1/2}(r-1))/\lambda^{1/2}$.

Applying the Schmidt process to the functions $\beta_1(r)$ and $\beta_2(r)$, where $\beta_1(r) = 1/f(r)$, $\beta_2(r) = g(r)/f(r)$, we obtain

$$\beta_1(r) = M_1 \alpha_1(r), \\ \beta_2(r) = M_2 \alpha_1(r) + M_3 \alpha_2(r),$$

where $\alpha_1(r)$ and $\alpha_2(r)$ are O.N. on $[0, 1]$, and

$$M_1^2 = \int_0^1 [\beta_1(r)]^2 dr, \\ M_2 = M_1^{-1} \int_0^1 \beta_1(r) \beta_2(r) dr, \\ M_3^2 = \int_0^1 [\beta_2(r) - M_2/M_1 \beta_1(r)]^2 dr.$$

On substituting, the right side of (14) becomes

$$(15) \quad (\sec \lambda^{1/2})^{1/2} \int_C F \left[- \int_0^1 (M_2 \alpha_1(r) + M_3 \alpha_2(r)) dy(r), \int_0^1 M_1 \alpha_1(r) dy(r) \right] d_w y.$$

Finally, an application of (13) to (15) gives

$$\begin{aligned} & \int_C \exp \left\{ \lambda \int_0^1 [x(s)]^2 ds \right\} F \left(\int_0^1 x(s) ds, x(1) \right) d_w x \\ &= \pi^{-1} (\sec \lambda^{1/2})^{1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(-M_3 u_2 - M_2 u_1, M_1 u_1) \\ & \quad \cdot \exp \left\{ -u_1^2 - u_2^2 \right\} du_1 du_2. \end{aligned}$$

Where a factor of the form $\exp \{ \mu [x(s)]^2 \}$ occurs in an integrand, it suffices for our purpose to note that it follows from a theorem of Erdős and Kac [4] that this functional is integrable over C for $\mu < 1$.

Using the above-mentioned facts and conditions (9) and (10), Cameron's proofs can quite readily be modified to give proofs for the sharpened theorems. The extension of Theorem 1 to n -dimensions is not difficult.

As these proofs are lengthy and as there are no essential difficulties, they will not be written out.

REFERENCES

1. R. H. Cameron, *The generalized heat flow equation and a corresponding Poisson formula*, Ann. of Math. vol. 59 (1954) pp. 434-462.
2. R. H. Cameron and R. E. Fagen, *Nonlinear transformations of Volterra type in Wiener space*, Trans. Amer. Math. Soc. vol. 75 (1953) pp. 552-575.
3. R. H. Cameron and W. T. Martin, *The transformation of Wiener integrals by nonlinear transformations*, Trans. Amer. Math. Soc. vol. 66 (1949) pp. 253-283.
4. P. Erdős and M. Kac, *On certain limit theorems of the theory of probability*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 292-302.
5. M. Rosenblatt, *On a class of Markoff processes*, Trans. Amer. Math. Soc. vol. 71 (1951) pp. 120-135.
6. N. Wiener, *Generalized harmonic analysis*, Acta Math. vol. 55 (1930).

DALHOUSIE UNIVERSITY