ON A SEQUENCE OF FOURIER COEFFICIENTS

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1. Let $f(t)$ be a function which is integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$ and is defined outside this interval by periodicity. Let the Fourier series of $f(x)$ be

$$(1.1) \quad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2} a_0 + \sum_1^{\infty} A_n(x);$$

then the conjugate series of (1.1) is

$$(1.2) \quad \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_1^{\infty} B_n(x).$$

Fejèr [3, p. 62] has shown that if $l = f(x+0) - f(x-0)$ exists and is finite, the sequence $\{nB_n(x)\}$ is summable (C, r) , $r > 1$, to the value l/π ; and if f is of bounded variation, the theorem holds true for $r > 0$. It has also been proved that if l exists and is finite, the sequence $\{nB_n(x)\}$ is summable by the first logarithmic mean to the same value [3, p. 62].

Obrechhoff [2] has shown that if f is integrable (L) and if $t^{-1}|f(x+t) - f(x-t) - l|$ is integrable near $t=0$, then $n^{-1} \sum_1^n rB_r(x) \rightarrow \pi^{-1}l$.

Mohanty and Nanda [1] have proved that if

$$f(x+t) - f(x-t) - l = o\{(\log 1/t)^{-1}\} \quad \text{as } t \rightarrow 0$$

and a_n and b_n are $O(n^{-\delta})$, $0 < \delta < 1$, then the sequence $\{nB_n(x)\}$ is summable $(C, 1)$ to the value l/π . From this result they have deduced the Hardy and Littlewood's test for convergence of the conjugate series (1.2) by applying a Tauberian theorem of Hardy and Littlewood.

We write

$$\psi(t) = f(x+t) - f(x-t) - l.$$

The object of this note is to prove the following theorem:

THEOREM 1. *If*

$$(1.3) \quad \Psi(t) = \int_0^t \psi(u) du = o(t)$$

and

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$$(1.4) \quad \int_{\epsilon}^{\delta} \frac{|\psi(t+\epsilon) - \psi(t)|}{t} dt \rightarrow 0,$$

for some fixed δ , when $\epsilon \rightarrow +0$, then the sequence $\{nB_n(x)\}$ is summable $(C, 1)$ to the value l/π .

From Theorem 1, we deduce Lebesgue's test for convergence of the conjugate series (1.2) by applying Tauber's Second Theorem.

It is to be noted that Obrechhoff has proved the summability $(C, 1)$ of the sequence $\{nB_n(x)\}$ under Dini's criterion for the convergence of Fourier series while from Fejèr's result it is obvious that the sequence is also summable $(C, 1)$ under Jordan's criterion. In Theorem 1, we prove the same result under Lebesgue's criterion. As it is well known that Lebesgue's test includes both Dini's and Jordan's tests, Theorem 1 includes the above mentioned results of Obrechhoff and Fejèr.

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2. We shall require the following lemmas:

LEMMA 1. If

$$I = \int_{\pi/n}^{\delta} \frac{\psi(t) \sin nt}{t^2} dt, \quad I' = \int_{\pi/n}^{\delta} \frac{\psi(t) \sin nt}{t(t + \pi/n)} dt,$$

and $\psi(t)$ satisfies (1.3), then

$$I - I' = o(n), \quad \text{as } n \rightarrow \infty.$$

We have

$$\begin{aligned} I - I' &= \frac{\pi}{n} \int_{\pi/n}^{\delta} \frac{\psi(t)}{t^2(t + \pi/n)} \sin ntdt \\ &= \frac{\pi}{n} \left[\frac{\Psi(\delta) \sin n\delta}{\delta^2(\delta + \pi/n)} - \int_{\pi/n}^{\delta} \Psi(t) \frac{d}{dt} \left\{ \frac{\sin nt}{t^2(t + \pi/n)} \right\} dt \right] \\ &= \frac{\pi}{n} \left[O(1) + o \left(\int_{\pi/n}^{\delta} t \left(\frac{n}{t^3} + \frac{1}{t^4} \right) dt \right) \right] \\ (2.1) \quad &= o(n) \quad \text{by (1.3),} \end{aligned}$$

which proves the lemma.

LEMMA 2. If $\psi(t)$ satisfies (1.3), then

$$\int_{\pi/n}^{2\pi/n} \frac{\psi(t)}{t} \cos ntdt = o(1) \quad \text{as } n \rightarrow \infty.$$

We have, by integration by parts,

$$\begin{aligned}\int_{\pi/n}^{2\pi/n} \frac{\psi(t)}{t} \cos ntdt &= \left[\Psi(t) \frac{\cos nt}{t} \right]_{\pi/n}^{2\pi/n} - \int_{\pi/n}^{2\pi/n} \Psi(t) \frac{d}{dt} \left(\frac{\cos nt}{t} \right) dt \\ &= o(1) + o \left(\int_{\pi/n}^{2\pi/n} \left(n + \frac{1}{t} \right) dt \right) \\ &= o(1) + o([\log t]_{\pi/n}^{2\pi/n}) \\ &= o(1).\end{aligned}$$

COROLLARY. If

$$\pi/n \leq \xi_1 < \xi_2 \leq 2\pi/n,$$

then

$$(2.2) \quad \int_{\xi_1}^{\xi_2} \psi(t) \frac{\cos nt}{t} dt = o(1)$$

and similarly

$$\int_{\xi_1}^{\xi_2} \psi(t) \frac{\sin nt}{t} dt = o(1).$$

LEMMA¹ 3. If $\psi(t)$ satisfies (1.3) and (1.4), then

$$(2.3) \quad \begin{aligned}J &= \int_{\pi/n}^{\delta} \frac{\psi(t)}{t(t + \pi/n)} e^{nit} dt \\ &= o(n).\end{aligned}$$

We have

$$(2.4) \quad \begin{aligned}J &= \left(\int_{\pi/n}^{2\pi/n} + \int_{2\pi/n}^{\delta} \right) \frac{\psi(t)}{t(t + \pi/n)} e^{nit} dt \\ &= J_1 + J_2,\end{aligned}$$

say. Now by (2.2)

$$(2.5) \quad \begin{aligned}J_1 &= \frac{n}{2\pi} \int_{\pi/n}^{\xi} \frac{\psi(t)}{t} e^{nit} dt, & \frac{\pi}{n} < \xi < \frac{2\pi}{n}, \\ &= o(n).\end{aligned}$$

Also

¹ The proof of this lemma is on the lines of Hardy and Rogosinsky [4] and is given only for the sake of completeness.

$$\begin{aligned}
 J_2 &= \int_{2\pi/n}^{\delta} \frac{\psi(t)}{t(t+\pi/n)} e^{nit} dt = - \int_{\pi/n}^{\delta-\pi/n} \frac{\psi(t+\pi/n)}{(t+\pi/n)(t+2\pi/n)} e^{nit} dt \\
 &= - \int_{\pi/n}^{\delta} \frac{\psi(t+\pi/n)}{(t+\pi/n)(t+2\pi/n)} e^{nit} dt \\
 &\quad + \int_{\delta-\pi/n}^{\delta} \frac{\psi(t+\pi/n)}{(t+\pi/n)(t+2\pi/n)} e^{nit} dt \\
 (2.6) &= - \int_{\pi/n}^{\delta} \frac{\psi(t+\pi/n)}{(t+\pi/n)(t+2\pi/n)} e^{nit} dt + o(1).
 \end{aligned}$$

From (2.3), (2.4), (2.5) and (2.6) we have

$$\begin{aligned}
 J &= \frac{1}{2} \int_{\pi/n}^{\delta} \left\{ \frac{\psi(t)}{t(t+\pi/n)} - \frac{\psi(t+\pi/n)}{(t+\pi/n)(t+2\pi/n)} \right\} e^{nit} dt + o(n) \\
 &= \frac{1}{2} \int_{\pi/n}^{\delta} \frac{\psi(t) - \psi(t+\pi/n)}{(t+\pi/n)(t+2\pi/n)} e^{nit} dt \\
 &\quad + \frac{\pi}{n} \int_{\pi/n}^{\delta} \frac{\psi(t)}{t(t+\pi/n)(t+2\pi/n)} e^{nit} dt + o(n) \\
 &= J_3 + J_4 + o(n), \text{ say.}
 \end{aligned}$$

Now,

$$|J_3| \leq \frac{n}{6\pi} \int_{\pi/n}^{\delta} \frac{|\psi(t+\pi/n) - \psi(t)|}{t} dt = o(n),$$

since $\psi(t)$ satisfies (1.4).

Again,

$$\begin{aligned}
 J_4 &= \frac{\pi}{n} \frac{\Psi(\delta) e^{ni\delta}}{\delta(\delta+\pi/n)(\delta+2\pi/n)} \\
 &\quad - \frac{\pi}{n} \int_{\pi/n}^{\delta} \Psi(t) \frac{d}{dt} \left\{ \frac{e^{nit}}{t(t+\pi/n)(t+2\pi/n)} \right\} dt \\
 &= O\left(\frac{1}{n}\right) + \frac{1}{n} \int_{\pi/n}^{\delta} o(t) \cdot O\left(\frac{n}{t^3} + \frac{1}{t^4}\right) dt \\
 &= o(1) + o(n) = o(n).
 \end{aligned}$$

Thus

$$(2.7) \quad J = o(n),$$

which proves the lemma.

3. Proof of the theorem. From Mohanty and Nanda [1], we have

$$\begin{aligned}
 \frac{1}{n} \sum_1^n r B_r(x) - \frac{l}{\pi} &= \frac{1}{\pi} \int_0^\pi \{f(x+t) - f(x-t) - l\} g(n, t) dt + o(1) \\
 &= \frac{1}{\pi} \int_0^\pi \psi(t) g(n, t) dt + o(1) \\
 (3.1) \quad &= \frac{1}{\pi} P + o(1),
 \end{aligned}$$

say, where

$$\begin{aligned}
 g(n, t) &= -\frac{1}{n} \frac{d}{dt} \{\cos t + \cos 2t + \cdots + \cos nt\} \\
 &= -\frac{1}{n} \frac{d}{dt} \left\{ \frac{\cos((n+1)/2)t \sin nt/2}{\sin t/2} \right\} \\
 &= -\frac{1}{2n} \frac{d}{dt} \left\{ \frac{\sin nt}{\tan t/2} + \cos nt - 1 \right\} \\
 &= \left\{ \frac{1}{4n} \frac{\sin nt}{\sin^2 t/2} - \frac{1}{2} \frac{\cos nt}{\tan t/2} \right\} + \frac{1}{2} \sin nt.
 \end{aligned}$$

Now

$$\begin{aligned}
 P &= \int_0^\pi \psi(t) g(n, t) dt \\
 &= \int_0^\pi \psi(t) \left[\frac{\sin nt}{4n \sin^2 t/2} - \frac{\cos nt}{2 \tan t/2} \right] dt + \frac{1}{2} \int_0^\pi \psi(t) \sin nt dt \\
 &= \int_0^\pi \psi(t) \left[\frac{\sin nt}{nt^2} - \frac{\cos nt}{t} \right] dt + o(1) \\
 &= \left(\int_0^{\pi/n} + \int_{\pi/n}^\delta + \int_\delta^\pi \right) \psi(t) \left[\frac{\sin nt}{nt^2} - \frac{\cos nt}{t} \right] dt + o(1) \\
 &= \left(\int_0^{\pi/n} + \int_{\pi/n}^\delta \right) \psi(t) \sigma(n, t) dt + o(1) \\
 (3.2) \quad &= P_1 + P_2 + o(1),
 \end{aligned}$$

say, by Riemann Lebesgue Theorem, where

$$\sigma(n, t) = \frac{\sin nt}{nt^2} - \frac{\cos nt}{t} = O(n^2 t), \quad \text{as } t \rightarrow 0.$$

Now,

$$\begin{aligned} P_1 &= [\Psi(t)\sigma(n, t)]_0^{\pi/n} - \int_0^{\pi/n} \Psi(t) \frac{d}{dt} \sigma(n, t) dt \\ (3.3) \quad &= o(1) + o\left(n^2 \int_0^{\pi/n} t dt\right) = o(1), \end{aligned}$$

by (1.3).

Again, by applying Lemma 1 and Lemma 2, we have

$$\begin{aligned} P_2 &= \frac{1}{n} \int_{\pi/n}^{\delta} \frac{\psi(t) \sin nt}{t^2} dt - \int_{\pi/n}^{\delta} \psi(t) \frac{\cos nt}{t} dt \\ &= \frac{1}{n} \int_{\pi/n}^{\delta} \frac{\psi(t) \sin nt}{t(t + \pi/n)} dt - \int_{\pi/n}^{\delta} \psi(t) \frac{\cos nt}{t} dt + o(1) \\ (3.4) \quad &= - \int_{\pi/n}^{\delta} \psi(t) \frac{\cos nt}{t} dt + o(1), \quad \text{by (2.7)} \\ &= - \left(\int_{\pi/n}^{2\pi/n} + \int_{2\pi/n}^{\delta+\pi/n} - \int_{\delta}^{\delta+\pi/n} \right) \psi(t) \frac{\cos nt}{t} dt + o(1) \\ &= - \int_{2\pi/n}^{\delta+\pi/n} \psi(t) \frac{\cos nt}{t} dt + o(1) \\ (3.5) \quad &= \int_{\pi/n}^{\delta} \frac{\psi(t + \pi/n) \cos nt}{t + \pi/n} dt + o(1). \end{aligned}$$

By (3.3) and (3.4) we have

$$\begin{aligned} |P_2| &= \frac{1}{2} \left| \int_{\pi/n}^{\delta} \left\{ \frac{\psi(t + \pi/n)}{t + \pi/n} - \frac{\psi(t)}{t} \right\} \cos ntdt \right| + o(1) \\ &= \frac{1}{2} \left| \int_{\pi/n}^{\delta} \frac{\psi(t + \pi/n) - \psi(t)}{t + \pi/n} \cos ntdt \right. \\ &\quad \left. - \frac{\pi}{2n} \int_{\pi/n}^{\delta} \frac{\psi(t)}{t(t + \pi/n)} \cos ntdt \right| + o(1) \\ &\leq \frac{1}{2} \int_{\pi/n}^{\delta} \frac{|\psi(t + \pi/n) - \psi(t)|}{t} dt \\ &\quad + \frac{\pi}{2n} \left| \int_{\pi/n}^{\delta} \frac{\psi(t)}{t(t + \pi/n)} \cos ntdt \right| + o(1) \\ (3.6) \quad &= o(1) + \frac{\pi}{2n} \cdot o(n) = o(1), \end{aligned}$$

by (2.7) and (1.4).

From (3.2), (3.3) and (3.6) we obtain

$$P = o(1).$$

Hence from (3.1) we have

$$\frac{1}{n} \sum_1^n r B_r(x) - \frac{l}{\pi} = o(1) \quad \text{as } n \rightarrow \infty,$$

which proves the theorem.

4. Lebesgue's criterion for the convergence of the conjugate series is
THEOREM 2. *If*

$$\theta(t) = f(x+t) - f(x-t)$$

satisfies the conditions

$$(4.1) \quad \int_0^t \theta(u) du = o(t)$$

and

$$(4.2) \quad \int_{\epsilon}^{\delta} \frac{|\theta(t+\epsilon) - \theta(t)|}{t} dt \rightarrow 0,$$

for some fixed δ , when $\epsilon \rightarrow +0$, then the conjugate series $\sum_1^{\infty} B_n(x)$ converges to the value

$$(4.3) \quad \frac{1}{2\pi} \int_{-0}^{\pi} \theta(t) \cot \frac{t}{2} dt$$

provided that the integral exists as a Cauchy integral at the origin.

We shall now deduce Theorem 2 as a corollary of Theorem 1 by employing the following

TAUBER'S SECOND THEOREM. *If $\sum u_n$ is summable (A), then a necessary and sufficient condition that it should be convergent is that the sequence $\{nu_n\}$ is summable (C, 1) to the value zero.*

Proof of Theorem 2. The existence of the integral (4.3) as a "Cauchy Integral" at the origin implies the summability (A) of the conjugate series $\sum_1^{\infty} B_n(x)$, [3, p. 55].

By using Theorem 1, we find that conditions (4.1) and (4.2) of Theorem 2 imply the summability (C, 1) of the sequence $\{nB_n(x)\}$ to the value zero. The convergence of the conjugate series then follows from Tauber's Second Theorem.

REFERENCES

1. R. Mohanty and M. Nanda, *On the behaviour of Fourier coefficients*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 79-84.
2. N. Obrechhoff, *Sur la série conjuguée de la série de Fourier*, Annuaire Université de Sofia. I. Faculté Physico-Mathématique. Livre 1, vol. 39 (1943) pp. 321-380.
3. A. Zygmund, *Trigonometrical series*, 1935, pp. 55, 62.
4. G. H. Hardy and W. W. Rogosinsky, *Fourier series*, 1944, p. 44.

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A NOTE ON HAUSDORFF TRANSFORMS

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1. **Introduction.** If $\{\mu_p\}$ ($p=0, 1, 2, \dots$) is an arbitrary sequence of numbers, we shall define the double sequence $(\Delta^n \mu_p)$ ($n, p=0, 1, 2, \dots$) by

$$(1.1) \quad \Delta^n \mu_p = \sum_{r=0}^n (-1)^r \binom{n}{r} \cdot \mu_{p+r} \quad (n, p = 0, 1, 2, \dots).$$

The quantities (1.1) are generally referred to as the "differences" of the sequence $\{\mu_p\}$ since they satisfy the recurrence relations

$$(1.2) \quad \begin{aligned} \Delta^0 \mu_p &= \mu_p & (p = 0, 1, 2, \dots), \\ \Delta^n \mu_p &= \Delta^{n-1} \mu_p - \Delta^{n-1} \mu_{p+1} & (n = 1, 2, 3, \dots; p = 0, 1, 2, \dots). \end{aligned}$$

In this paper, we shall say that the sequence $\{t_p\}$ is the Hausdorff transform of the sequence $\{s_p\}$ generated by the sequence $\{\mu_p\}$ in case

$$(1.3) \quad t_p = \sum_{q=0}^p \binom{p}{q} \cdot (\Delta^{p-q} \mu_q) \cdot s_q \quad (p = 0, 1, 2, \dots).$$

A Hausdorff transform generated by a sequence $\{\mu_p\}$ will be called here, for shortness, an (H, μ_p) transform.

Although transformations of the form (1.3) are generally referred to as Hausdorff transformations, it is to be noted that they were actually first investigated to any extent by Hurwitz and Silverman [1],¹ who characterized them as sequence-to-sequence transformations generated by triangular matrices A permutable with the matrix

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¹ Numbers in square brackets refer to references at the end of this note.