AN EXAMPLE IN THE PROBLEM OF LEAST AREA¹

WENDELL H. FLEMING

1. Introduction. The existence of a surface of least area bounded by a given simple closed curve C (or any finite number of disjoint simple closed curves) in euclidean 3-space R^3 , and of prescribed finite Euler characteristic and character of orientability is well known (see Courant [1], Douglas [2]; also Radó [5] for the simply connected case). The solution has been generalized by Douglas [3] to include certain prescribed infinite topological types as well. Little is known, however, about the case when the topological type is arbitrary rather than prescribed. In the special case when the boundary curve C has a simply covered orthogonal projection on the perimeter of some convex planar region B , the problem for unrestricted topological types has a solution which has a representation in nonparametric form on B and hence is a surface of the type of the disc. This can be proved using a symmetrization argument used by Garabedian and Schiffer [4, p. 445] and the existence of a solution of the least area problem for the type of the disc [1; 2; 5].

The object of the present note is to give an example of a simple closed rectifiable curve C in R^3 for which the problem of least area with unrestricted topological types has no solution of finite topological type.

The letter $x = (x^1, x^2, x^3)$ will denote a point in R^3 . $C, \Gamma, \dots$ will stand for a closed Fréchet curve in R^3 , nonoriented unless otherwise stated. If the curve is simple closed, the same letter will also be used to denote the point set it occupies. A curve C is termed *thin* if its set of points is of 2-dimensional Hausdorff measure 0.

Let $V_1, \dots, V_n$ be planar 2-cells, each bounded by a rectifiable simple closed curve (in the present applications the boundaries will actually be piecewise analytic). From $V_1, \dots, V_n$ we shall define a 2-manifold D with boundary by means of a finite number of identifications of boundary arcs in pairs, after the manner of Siefert-Threlfall [7, Chap. 6]. Identification of a pair of arcs γ_1 and γ_2 is realized by a homeomorphism of γ_1 onto γ_2 . Assume that D is connected and that the boundary D^* (union of the unidentified boundary arcs) is a simple closed curve. Let $D^0 = D - D^*$ denote the interior of D . By [7, p. 142],

Presented to the Society, May 1, 1954; received by the editors August 19, 1954 and, in revised form, December 13, 1954 and October 24, 1955.

¹ This paper was sponsored by the Office of Ordnance Research, U. S. Army.

D is topologically a disc with a finite number h handles (orientable case) or q cross-caps (nonorientable case). The topological type τ of D is specified by its orientability or nonorientability and characteristic r , where $r = 2h - 1$ or $q - 1$ according as D is orientable or non-orientable.

Let w denote a point of D , and $x(w)$ a continuous mapping from D to R^3 with the following property:

(1.1) For each $i = 1, \dots, n$, $x(w)$ agrees in V_i with a mapping $x_i(u, v)$ continuous on V_i and satisfying a Lipschitz condition on each closed subset of V_i^0 .

Let

$$a(x, D) = \sum_{i=1}^n \iint_{V_i^0} [EG - F^2]^{1/2} du dv,$$

$$I(x, D) = \sum_{i=1}^n \iint_{V_i^0} \frac{1}{2} (E + G) du dv$$

denote, respectively, the classical area integral and the Dirichlet integral, where E , F , and G have their usual meaning in differential geometry. The quantities $a(x, D)$ and $I(x, D)$ are invariant under subdivision and conformal transformation of the cells V_i .

If $V_1, \dots, V_n$ are triangles and $x_i(u, v)$ is one-one and linear on each V_i , then $x(w)$ defines a (Fréchet) *polyhedron* P of the same type as D . On D^* , $x(w)$ represents a closed polygonal curve C , termed the *boundary* of P . Let $e(P)$ denote the elementary area of P . For such a piecewise linear representation $x(w)$ of P , $e(P) = a(x, D)$.

Given a simple closed curve C in R^3 and τ , let

$$m(C, \tau) = \inf_{\{P_n\}} \liminf_n e(P_n),$$

where P_n is a polyhedron of type τ , $n = 1, 2, \dots$, whose boundary C_n tends to C in the Fréchet sense. $m(C, \tau)$ is termed the *least area for type τ and boundary C* , and agrees with the inf of the Lebesgue area for surfaces of type τ and boundary C [2, pp. 295–296]. By *least area for unrestricted topological type and boundary C* we mean

$$m(C) = \inf_{\{P_n\}} \liminf_n e(P_n),$$

where as before the boundary C_n of P_n tends to C but now the type of P_n is arbitrary.

(1.2) LEMMA. Let C be a closed, rectifiable Fréchet curve in R^3 . Let $x(w)$ be a continuous mapping from a domain D of type τ , defined as

above by planar 2-cells $V_1, \dots, V_n$, such that:

- (a) (1.1) holds;
- (b) $I(x, D)$ is finite;
- (c) $x(w)$ represents C on D^* ;
- (d) for each $i=1, \dots, n$, the mapping $x_i(u, v)$ agreeing with $x(w)$ in V_i represents a thin curve on V_i^* .

Then there exists a sequence P_n of polyhedra of type τ whose boundaries C_n tend to C in the Fréchet sense such that $\lim_n e(P_n) = a(x, D)$.

By making conformal transformations, we may suppose all V_i are polygons. Lemma (1.2) is then essentially a special case of approximation theorems recently published by Young [8] in his treatment of the general problem of minimum for parametric surfaces of prescribed, finite topological type (see especially [8, (2.3), (3.2), (4.2), (6.5)]). Young's methods actually yield a more general form of (1.2), obtained by replacing pointwise identification along identified boundary arcs by a less restrictive type of boundary identification.

We shall need the fundamental existence theorem [1; 2] for the problem of least area with prescribed type τ , which is quoted as (1.3) below. The domain D appearing in the solution is of a particular normal form, and is defined somewhat differently than above. For definiteness we shall follow Courant [1], where the normal domain is a half-plane domain with a certain finite set of identified slits parallel to the real axis. To represent a half-plane slit domain in the form above, defined by a finite number of planar 2-cells $V_1, \dots, V_n$, first transform D into a bounded domain by inverting about a point in the complementary half-plane, and then make a suitable dissection. Given a boundary point $w^* \in D^*$, we may further arrange (by means of elementary conformal transformations, reflections, subdivision, and combining of cells) that V_1 is the upper half K of the unit circle and that w^* corresponds to the point $(0, 0)$ of K .

For such normal domains, D^0 has an analytic as well as a topological structure. In particular, harmonic functions can be defined in D^0 ($x(w)$ is harmonic in D^0 if near each point $w^0 \in D^0$ the components $x^j(w)$, $j=1, 2, 3$, are harmonic when referred to a set of local variables (u, v) [1, p. 76]).

(1.3) Let C be a simple closed curve in R^3 such that $m(C, \tau)$ is finite, and either: (i) τ is the type of the disc, or (ii) τ is some higher type and $m(C, \tau) < m(C, \tau')$ for any τ' of characteristic smaller than that of τ . Then there exists a normal domain D of type τ and $x(w)$ continuous on D , such that:

- (a) $x(w)$ is harmonic in D^0 ;
- (b) $E=G, F=0$ in D^0 ;

- (c) $a(x, D) = I(x, D) = m(C, \tau)$;
 (d) $x(w)$ maps D^* one-one onto C .

We shall consider here only rectifiable curves C , for which the hypothesis $m(C, \tau)$ finite holds for any τ . Items (1.3a) and (1.3b) express that $x(w)$ is a minimal surface.

2. The example. We begin with an example due to Douglas [2, p. 209] of a simple closed curve Γ_0 for which the minimum area for surfaces of the type of the disc is much greater than the absolute minimum area. The equations

$$\begin{aligned}x^1 &= -1 + \cos \theta(2 - \cos \phi), \\x^2 &= \sin \theta(2 - \cos \phi), \\x^3 &= \sin \phi, \quad -\pi \leq \theta, \phi \leq \pi,\end{aligned}$$

define a torus T with center $(-1, 0, 0)$, major radius 2, and minor radius 1. Let us denote this mapping by $f(\theta, \phi)$. For given fixed δ , $0 < \delta < 2^{-5}$, let

$$S_1 = \left[(\theta, \phi) \mid |\theta| \leq \frac{\delta}{2} \right], \quad S_2 = \left[(\theta, \phi) \mid |\phi| \leq \frac{\delta}{2} \right].$$

Let S denote the domain obtained from $S_1 \cup S_2$ upon identifying $(-\pi, \phi)$ with (π, ϕ) and $(\theta, -\pi)$ with (θ, π) . S is of the type of the disc with one handle, and $f(\theta, \phi)$ maps S onto a portion of T of area $< 7\pi\delta$. The image of S^* is a simple closed curve, which we denote by Γ_0 . For simply connected surfaces bounded by Γ_0 , the area is at least $2\pi(1-\delta)^2$. This is a special case of (2.1) below.

For each pair of real numbers ρ_1, ρ_2 , $\rho_1 < \rho_2$, let

$$B(\rho_1, \rho_2) = [x \in R^3 \mid \rho_1 < x^1 < \rho_2].$$

(2.1) LEMMA. Let Γ be a simple closed curve in R^3 such that $\Gamma \cap B(-2+\delta, 2-\delta) = \Gamma_0 \cap B(-2+\delta, 2-\delta)$. Let D be a planar 2-cell bounded by a simple closed curve and $x(u, v)$ a mapping from D to R^3 such that:

- (a) $x(u, v)$ is continuous on D and Lipschitzian on each closed subset of D^0 ;
 (b) $a(x, D)$ is finite;
 (c) On D^* , $x(u, v)$ represents Γ .

Then $a(x, D) \geq 2\pi(1-\delta)^2$.

PROOF. We may suppose D a square. Let $0 < \epsilon < \delta$. By a known theorem there exists a piecewise linear mapping $y(u, v)$ such that $|y(u, v) - x(u, v)| < \epsilon$ for $(u, v) \in D$ and $a(y, D) < a(x, D) + \epsilon$. Let

$(u_1, v_1), \dots, (u_4, v_4)$ be four points of D^* such that $x^1(u_i, v_i) = (-1)^{i+1}$, $i=1, 2, 3, 4$, and $(u_1, v_1) \cup (u_3, v_3)$ separates (u_2, v_2) and (u_4, v_4) in D^* . Let

$$K = [(u, v) \in D \mid y^1(u, v) = 0].$$

Either K separates (u_1, v_1) and (u_3, v_3) in D , or K separates (u_2, v_2) and (u_4, v_4) in D . For if not, then $(u_1, v_1), (u_3, v_3)$ can be joined in $D-K$ by a polygonal arc π_1 , and $(u_2, v_2), (u_4, v_4)$ can be joined in $D-K$ by a polygonal arc π_2 . $\pi_1 \cap \pi_2$ is nonempty since $(u_1, v_1) \cup (u_3, v_3)$ separates (u_2, v_2) and (u_4, v_4) in D^* . However, $y^1(u, v) > 0$ on π_1 and $y^1(u, v) < 0$ on π_2 , a contradiction.

For definiteness, suppose (u_1, v_1) and (u_3, v_3) are separated by K , the remaining case being similar. Since $y(u, v)$ is piecewise linear, (u_1, v_1) and (u_3, v_3) are then separated by a polygonal arc $\pi \subset K$. We may suppose $\pi \cap D^*$ consists of only two points. Then $D^* - \pi$ consists of two disjoint arcs γ_1, γ_2 with $(u_1, v_1) \in \gamma_1, (u_3, v_3) \in \gamma_2$. On γ_1, γ_2 , respectively, $x(u, v)$ represents nonoverlapping arcs Γ', Γ'' of Γ whose endpoints are situated in $B(-\delta, \delta)$ since $\epsilon < \delta$. If we suppose for definiteness that $x^2(u_1, v_1) > 0$, then

$$\Gamma' \cap B(\delta, 2 - \delta) = [x \in \Gamma \cap B(\delta, 2 - \delta) \mid x^2 > 0],$$

$$\Gamma'' \cap B(\delta, 2 - \delta) = [x \in \Gamma \cap B(\delta, 2 - \delta) \mid x^2 < 0].$$

If $Y(p)$ is a continuous mapping of an oriented simple closed planar curve ζ into a plane A , let $\mu(q, Y, \zeta)$ denote as in [6] the topological index with respect to a point $q \in A$; for any arc η of ζ on which $Y(p) \neq q$, let $V_\eta [\arg (Y(p) - q)]$ denote the variation in argument of $Y(p) - q$ on η . The following known properties hold (cf. [6, esp. pp. 142-149]):

(1) Let $\zeta = \eta_1 \cup \dots \cup \eta_n$, where $\eta_1, \dots, \eta_n$ are pairwise nonoverlapping arcs. If $Y(p) \neq q$ on ζ , then

$$\mu(q, Y, \zeta) = \frac{1}{2\pi} \sum_{i=1}^n V_{\eta_i} [\arg (Y(p) - q)].$$

(2) Let l be a straight line not containing q . Let $Y(p)$ be in the closed half-space bounded by l not containing q , for all $p \in \eta$, and be on l at the end points of η . Then $V_\eta [\arg (Y(p) - q)]$ equals the angle $Y(p_1)qY(p_2)$, where p_1 is the initial end point of η and p_2 the final.

(3) $\mu(q, Y_1, \zeta) = \mu(q, Y_2, \zeta)$, if

$$\sup_{p \in \zeta} |Y_1(p) - Y_2(p)| < \inf_{p \in \zeta} |Y_1(p) - q|.$$

(4) If D is a planar 2-cell, $Y(p)$ is defined and continuous on D , and $\mu(q, Y, D^*) \neq 0$, then $q \in Y(D)$.

Let D_1 denote the 2-cell bounded by $\gamma_1 \cup \pi$. Let $X(u, v)$, $Y(u, v)$ denote the plane mappings obtained from $x(u, v)$, $y(u, v)$, respectively, upon setting the second coordinate x^2 equal to 0. Clearly, $|X(u, v) - Y(u, v)| < \epsilon$, $(u, v) \in D$.

Let $\Gamma^\wedge$ denote the ellipse

$$x^1 = -1 + \cos \frac{\delta}{2} (2 - \cos \phi), \quad x^2 = 0, \quad x^3 = \sin \phi, \quad -\pi \leq \phi \leq \pi.$$

The circular disc $(x^1 - 1)^2 + (x^3)^2 \leq (1 - \delta)^2$ in the plane $x^2 = 0$ is inside $\Gamma^\wedge$. $X(u, v)$ represents the part of $\Gamma^\wedge$ for which $\delta < x^1 < 2 - \delta$ on a pair of open arcs of D_1^* with disjoint closures, and $X^1(u, v)$ is outside this interval on the complementary pair of arcs of D_1^* . If $q = (x^1, x^3)$ and $(x^1 - 1)^2 + (x^3)^2 < (1 - \delta)^2$, then $|\mu(q, X, D_1^*)| = 1$ by (1) and (2) (let us assign a definite orientation to D_1^* as required in the definition of index μ). If $(x^1 - 1)^2 + (x^3)^2 < (1 - \delta - \epsilon)^2$, then $|\mu(q, Y, D_1^*)| = 1$ by (3). By (4) and the fact that $Y(u, v)$ is piecewise linear,

$$a(Y, D_1) \geq \pi(1 - \delta - \epsilon)^2.$$

Applying a similar argument to the 2-cell D_2 bounded by $\gamma_2 \cup \pi$, $a(Y, D_2) \geq \pi(1 - \delta - \epsilon)^2$. Since Y is the projection of y , and y is piecewise linear,

$$a(y, D) = a(y, D_1) + a(y, D_2) \geq a(Y, D_1) + a(Y, D_2) \geq 2\pi(1 - \delta - \epsilon)^2.$$

Since ϵ is arbitrary, this proves (2.1).

Let $U_k(x)$ denote the similarity transformation defined by

$$U_k(x) = 2^{-k}x + a_k, \\ a_k = [2\delta^{-1}(1 - 2^{-k}), 0, 0], \quad k = 0, 1, 2, \dots$$

Let $\Gamma_k = U_k(\Gamma_0)$; the curves Γ_k shrink down to a point $x^* = (2\delta^{-1}, 0, 0)$. Let α_0 denote an open arc of Γ_0 of length $< \pi\delta^2/4$ and midpoint

$$f\left(\frac{\delta}{2}, \pi\right) = \left(-1 + 3\cos\frac{\delta}{2}, 3\sin\frac{\delta}{2}, 0\right).$$

Let β_0 denote an open arc of Γ_1 of length $< \pi\delta^2/4$ with midpoint

$$U_1\left[f\left(\pi, \frac{\delta}{2}\right)\right] = \left(-\frac{3}{2} + \frac{1}{2}\cos\frac{\delta}{2} + \delta^{-1}, 0, \frac{1}{2}\sin\frac{\delta}{2}\right).$$

Let d_0 and e_0 denote a pair of line segments whose left (right) end

points are the ends of $\alpha_0(\beta_0)$. $d_0 \cap e_0$ is empty. Let

$$(2.2) \quad C = \{x^*\} \cup \left[\bigcup_{k=0}^{\infty} U_k(\Gamma_0 \cup d_0 \cup e_0) - \bigcup_{k=0}^{\infty} U_k(\alpha_0 \cup \beta_0) \right].$$

C is a rectifiable simple closed curve, and has the desired property:

(2.3) THEOREM. *For the rectifiable simple closed curve C defined by (2.2), $m(C, \tau) > m(C)$ for every finite topological type τ (no surface of finite type provides least area for unrestricted types).*

REMARK. The only really essential property of Γ_0 used in constructing C is that the absolute minimum area for Γ_0 is much less than the minimum area for surfaces of the type of the disc bounded by Γ_0 . As pointed out by R. H. Bing, there exists a simple closed curve Γ with this property situated in the union of two planes. For example, let $\Gamma = \Sigma_1 \cup \dots \cup \Sigma_5$, where

$$\Sigma_1 = [x \mid (x^1)^2 + (x^3)^2 = 1, x^3 \geq 0, x^2 = 0],$$

$$\Sigma_2 = [x \mid (x^1)^2 + (x^3)^2 = (1 - \delta)^2, x^3 \geq 0, x^2 = 0],$$

$$\Sigma_3 = [x \mid (x^1 + \delta/2)^2 + (x^2)^2 = (1 - \delta/2)^2, x^2 \geq 0, x^3 = 0],$$

$$\Sigma_4 = [x \mid (x^1 + \delta/2)^2 + (x^2)^2 = (1 - 3\delta/2)^2, x^2 \geq 0, x^3 = 0],$$

$$\Sigma_5 = [x \mid (x^1 - 1 + \delta)^2 + (x^2)^2 = \delta^2, x^2 \leq 0, x^3 = 0].$$

By a construction similar to the one above, we obtain a simple closed rectifiable curve C situated in the union of the planes $x^2 = 0$ and $x^3 = 0$ for which (2.3) holds. Professor Bing also suggested certain important simplifications in an earlier version of the proof of (2.1).

PROOF OF (2.3). We prove (2.3) by contradiction. Since C is rectifiable, $m(C)$ is finite; moreover, by definition, $m(C, \tau) \geq m(C)$ for all τ . Suppose $m(C, \tau) = m(C)$ for some finite type τ , in particular for some τ_0 of smallest characteristic. Then C and τ_0 satisfy the hypotheses of (1.3); therefore, there exist a normal domain D of type τ_0 and $x(w)$ defined in D such that (1.3a)–(1.3d) hold.

Let us first note some geometric properties satisfied by C . Let

$$b_k = a_k^1 + 2^{-k+1},$$

$$b'_k = a_{k+1}^1 - (1 + \delta)2^{-k}, \quad k = 0, 1, \dots$$

For $b_k \leq s \leq b'_k$, the plane $x^1 = s$ separates Γ_k and Γ_{k+1} , and meets C in a pair of points $x_s, \bar{x}_s$ interior to the line segments $d_k = U_k(d_0)$, $e_k = U_k(e_0)$, respectively.

Let L denote the solid right circular cone in R^3 with vertex x^* , defined by

$$[(x^2)^2 + (x^3)^2]^{1/2} \leq \frac{\delta}{2} \frac{2\delta^{-1} - x^1}{1 - \delta}, \quad x^1 \leq 2\delta^{-1}.$$

We show that $C \subset L$; to do this it will suffice to show that all $\Gamma_k \subset L$. Consider the case $k=0$ and set $\theta = \delta/2$ in the definition of Γ_0 . It suffices to consider $\pi/2 \leq \phi \leq 3\pi/4$, for which values of ϕ , $\sin^2 \phi \geq 1/2$. Then

$$\begin{aligned} (x^2)^2 + (x^3)^2 &= \sin^2 \frac{\delta}{2} (2 - \cos \phi)^2 + \sin^2 \phi \\ &\leq \sin^2 \phi \left(1 + \frac{9}{2} \delta^2 \right). \end{aligned}$$

Let $\rho = \delta(1 - \delta)^{-1/2}/2$. Then in a (ξ, η) plane, the circle $(\xi - 1)^2 + \eta^2 = 1$ is tangent to the line $\eta = \rho(2\delta^{-1} - \xi)$. Setting $\xi = 1 - \cos \phi$, $\eta = \sin \phi$, then $\sin \phi \leq \rho[2\delta^{-1} - (1 - \cos \phi)]$, whence

$$[(x^2)^2 + (x^3)^2]^{1/2} \leq \rho \left(1 + \frac{9}{2} \delta^2 \right)^{1/2} [2\delta^{-1} - (1 - \cos \phi)].$$

Since $1 - \cos \phi \geq -1 + (\cos \delta/2)(2 - \cos \phi) = x^1$, and $(1 + 9\delta^2/2)^{1/2} \cdot (1 - \delta)^{1/2} \leq 1$ for $0 < \delta < 2^{-5}$,

$$[(x^2)^2 + (x^3)^2]^{1/2} \leq \frac{\delta}{2} \frac{2\delta^{-1} - x^1}{1 - \delta}.$$

Thus $x = (x^1, x^2, x^3) \in L$. The cases $\theta = -\delta/2$, $\phi = \pm \delta/2$, and $k=1, 2, \dots$ are similar. Thus, $C \subset L$. Since L is a closed convex region containing C and $x(w)$ is harmonic in D^0 , the values $x(w)$ must lie in L for all $w \in D$ by the maximum modulus principle for harmonic functions.

For each s , the intersection of L with the plane $x^1 = s$ is a circular disc whose radius we denote by r_s . One verifies that for $s = b'_k$, $r_s < 2^{-k-1}(1 + 3\delta)$, $k=0, 1, \dots$.

By (1.3d), $x^* = x(w^*)$ for exactly one $w^* \in D^*$. Let D be represented by 2-cells $V_1, \dots, V_n$ as in §1. By (1.3a), the image of each boundary V_i^* is a finite or countable sum of rectifiable arcs and points, and hence is thin. By a previous remark, we may suppose that V_1 is the upper half circle K and that w^* corresponds to $(0, 0)$. The function $x_1(u, v)$ is continuous in K ; in K^0 , $x_1(u, v)$ is a minimal surface. For brevity, let us drop the subscript 1. Let λ denote the straight portion of K^* . By the reflection principle [1, p. 119], $x(u, v)$ has an analytic extension as a minimal surface across any arc of λ which maps onto an arc interior to some segment d_k or e_k of C .

By construction of C and the maximum modulus principle, $x^1(w) < 2\delta^{-1}$ for $w \neq w^*$. Hence, $x^1(u, v)$ has an upper bound $s_0 < 2\delta^{-1}$ on $K^* - \lambda$. Let k_0 be such that $s_0 < b_{k_0}$; for $b_k \leq s \leq b'_k$, $k > k_0$, let

$$W_s = [(u, v) \in K \mid x^1(u, v) = s].$$

$W_s \cap K^*$ consists of two points $(u_s, 0)$, $(\bar{u}_s, 0)$ in λ , with $x_s = x(u_s, 0)$, $\bar{x}_s = x(\bar{u}_s, 0)$.

Let $K^\wedge$ denote K^0 plus the union of all open arcs of λ across which $x(u, v)$ has an analytic extension. $W_s \subset K^\wedge$ for $s \in (b_k, b'_k)$, $k > k_0$. Since $x^1(u, v)$ is harmonic and not constant, the equation $x_u^1 = x_v^1 = 0$ has only isolated solutions in $K^\wedge$. Moreover, for all (u, v) ,

$$E = \frac{1}{2} (E + G) \geq \frac{1}{2} [(x_u^1)^2 + (x_v^1)^2].$$

Suppose $(x_u^1)^2 + (x_v^1)^2 > 0$ on W_s . Then W_s is an analytic arc; i.e., W_s has a representation $u = u(t)$, $v = v(t)$, $0 \leq t \leq 1$, which is one-one and analytic from $(0, 1)$ to W_s such that $(u')^2 + (v')^2 > 0$. Since $x^2(u, v)$ and $x^3(u, v)$ are analytic in (u, v) , $\phi(t) = x^2[u(t), v(t)]$ and $\psi(t) = x^3[u(t), v(t)]$ are analytic; by an easy calculation

$$(\phi')^2 + (\psi')^2 = E[(u')^2 + (v')^2] > 0.$$

Thus, the image of W_s is likewise an analytic arc, provided $x(u, v)$ is one-one on W_s .

I assert that for each $k > k_0$, $x(u, v)$ is one-one on W_s for a dense set of $s \in (b_k, b'_k)$. If this fails to hold for some k , then there is a sub-interval Δ of (b_k, b'_k) such that, for $s \in \Delta$, $(x_u^1)^2 + (x_v^1)^2 > 0$ on W_s and the equation $x(u, v) = x(\bar{u}, \bar{v})$ has a solution for distinct points (u, v) , $(\bar{u}, \bar{v})$ of W_s . Since $x(u, v)$ is analytic in (u, v) , it then follows from a known theorem that distinct analytic arcs γ_1 and γ_2 in $K^\wedge$ exist on which $x(u, v)$ describes the same curve. This contradicts a remark of Courant [1, p. 159] that a minimal surface providing absolute least area can have no line of self-intersection.²

Thus, for each $k > k_0$, there is a dense subset J_k of (b_k, b'_k) for which W_s and its image are analytic arcs. Let us now fix $k > k_0 + 1$ and $s \in J_{k-1}$ close enough to b'_{k-1} that $r_s < 2^{-k}(1 + 3\delta)$. Let l denote the line segment joining x_s and $\bar{x}_s$ and γ the analytic arc $x(W_s)$. We exclude

² The form of Courant's remark we want here is the following: Let $x(w)$ satisfy (1.3a)–(1.3d) relative to the normal domain D and simple closed curve C . Let there exist two analytic arcs γ_1 and γ_2 situated in D^0 , or D^0 plus an arc of D^* across which $x(w)$ has an analytic extension, on which $E \neq 0$ and $x(w)$ describes the same curve. Then $a(x, D) > m(C)$. Courant gives only a heuristic proof of the remark; however the details are elementary.

the case $l = \gamma$, which can be dealt with by a simplification of the argument below. Since l and γ are analytic arcs, $l \cap \gamma$ is a finite set of points $x_0, x_1, \dots, x_n$, ordered from x_s to $\bar{x}_s$ following the order of l , with $x_0 = x_s, x_n = \bar{x}_s$.

Let H denote the plane $x^1 = s$. The components of $H - (l \cup \gamma)$ can be colored "black" and "white" in such a way that any two components with a common boundary arc have opposite colors. The proof of this is elementary, and will merely be sketched. First, it suffices to replace γ by a suitable polygonal arc γ_1 likewise meeting l in $x_0, \dots, x_n$ and precisely those points. Next, we may eliminate the case where γ_1 touches l at some $x_i, 1 \leq i \leq n-1$, but does not cross l at x_i , by a suitable slight deformation of γ_1 near x_i . We then proceed by induction on n . The cases $n=1, 2$ are trivial. For $n > 2$ it must be true that for some $i, 1 \leq i \leq n-2, x_i$ and x_{i+1} are joined by an arc ζ of γ_1 which does not meet l between its end points. Let ζ' denote a suitable slightly larger arc of γ_1 , obtained by extending ζ past x_i and x_{i+1} along γ_1 . Replace ζ' by a line segment joining its end points. There are now $n-2$ intersections; hence, we can color. From this coloring it is easy to define a coloring for the original configuration.

For definiteness, color the unbounded component white. Let $Z_1, \dots, Z_\mu$ denote the black components plus their boundaries. Each Z_i is a 2-cell bounded by a finite number of analytic arcs, and each point of $l \cup \gamma$ is in at least one Z_i^* . $Z_1 \cup \dots \cup Z_\mu \subset H \cap L$, a circle of radius r_s .

Let us now replace k by $k+1$ and define $s', l', \gamma', Z'_1, \dots, Z'_\nu$ correspondingly. We have upon letting X_i, X'_j denote identity maps:

$$\sum_{i=1}^{\mu} a(X_i, Z_i) \leq \pi r_s^2 < \pi 2^{-2k} (1 + 3\delta)^2,$$

$$\sum_{j=1}^{\nu} a(X'_j, Z'_j) \leq \pi r_{s'}^2 < \pi 2^{-2(k+1)} (1 + 3\delta)^2 = \pi 2^{-2k} \frac{(1 + 3\delta)^2}{4}.$$

Let $f_k = U_k f$ (cf. definition of the torus T).

$$a(f_k, S) < \pi 2^{-2k} (7\delta).$$

In addition, by construction of C , there exist Lipschitzian one-one mappings $g(u, v), g'(u, v)$ from squares Z, Z' , respectively, such that: (1) on Z^* , g represents the curve made up of β_{k-1}, l , and the parts of d_{k-1} and e_{k-1} with $x^1 > s$, while g' represents on $(Z')^*$ the curve made up of α_k, l' , and the parts of d_k and e_k with $x^1 < s'$; (2) $a(g, Z) < \pi 2^{-2(k-1)} (\delta/4), a(g', Z') < \pi 2^{-2k} (\delta/4)$.

Let A denote the component of $K - W_s$ containing $(0, 0)$, A' being

defined similarly when s' replaces s . Since $s' > s$, $A' \subset A$. $Q = (A - A') \cup W_s$ is a 2-cell. On Q^* , $x(u, v)$ describes a simple closed curve C_1 whose image under the similarity transformation U_k^{-1} satisfies the hypotheses of (2.1). Hence, by (2.1),

$$a(x, Q) \geq \pi 2^{-2k} [2(1 - \delta)^2].$$

Consider the following collection: (1) $D - Q^0$; (2) $Z_1, \dots, Z_\mu$; (3) $Z'_1, \dots, Z'_\nu$; (4) S ; (5) Z ; (6) Z' . For $i = 1, \dots, \mu$, let $p \in Z_i^*$ be identified with $x^{-1}(p) \in W_s$ if $p \in \gamma$ and is not an isolated point of $\gamma \cap Z_i^*$, and with $g^{-1}(p) \in Z^*$ otherwise (similar boundary identifications for $Z'_j, j = 1, \dots, \nu$). In addition, identify each pair of points of S^* and Z^* (S^* and $(Z')^*$) corresponding to the same point of $\beta_{k-1}(\alpha_k)$. Let D_1 denote the domain obtained from (1)–(6) under these identifications, and $x_1(w)$ the mapping from D_1 which agrees with x, X_i, X'_j, f_k, g , and g' in their respective domains of definition.

D_1 is a domain of the kind defined in §1. To see this we need only verify that each identified arc is identified with just one other, D is connected, and the unidentified boundary arcs form a simple closed curve (cf. [7]). Moreover, $x_1(w)$ and C satisfy the hypotheses of (1.2) on D_1 .

Now $a(x, D - Q^0) = a(x, D) - a(x, Q) = m(C) - a(x, Q)$. Hence, recalling that $0 < \delta < 2^{-5}$,

$$\begin{aligned} a(x_1, D_1) &= a(x, D - Q^0) + \sum_{i=1}^{\mu} a(X_i, Z_i) + \sum_{j=1}^{\nu} a(X'_j, Z'_j) \\ &\quad + a(f_k, S) + a(g, Z) + a(g', Z') \\ &\leq m(C) + \pi 2^{-2k} \left[-2(1 - \delta)^2 + \frac{5}{4} (1 + 3\delta)^2 + 7\delta + \frac{5\delta}{4} \right] \\ &= m(C) + \frac{\pi}{4} 2^{-2k} [-3 + 79\delta + 37\delta^2] = m(C) + \sigma, \end{aligned}$$

where $\sigma < 0$. By (1.2), there exists a polyhedron P with boundary arbitrarily close to C such that $e(P) < m(C) + \sigma/2$, contradicting the definition of $m(C)$. This proves (2.3).

REFERENCES

1. R. Courant, *Dirichlet's principle, conformal mapping, and minimal surfaces*, Interscience, 1950.
2. J. Douglas, *Minimal surfaces of higher topological structure*, Ann. of Math. vol. 40 (1939) pp. 205–298.

* I.e., $V_1 - Q^0, V_2, \dots, V_n$.

3. ———, *Green's function and the problem of Plateau. The most general form of the problem of Plateau*, Amer. J. Math. vol. 61 (1939) pp. 545–608.
4. P. Garabedian, and M. Schiffer, *On a double integral variational problem*, Canadian Journal of Mathematics vol. 6 (1954) pp. 441–446.
5. T. Radó, *On the problem of Plateau*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 2, Springer, 1933.
6. ———, *Length and area*, Amer. Math. Soc. Colloquium Publications, vol. 30, 1948.
7. H. Seifert, and W. Threlfall, *Lehrbuch der Topologie*, Teubner, 1934.
8. L. C. Young, *On generalized surfaces of finite topological types*, Memoirs of the American Mathematical Society, no. 17, 1955.

RAND CORPORATION

AN ADDENDUM TO "A NOTE ON SUBSEQUENCES"

R. CREIGHTON BUCK

Some years ago, I obtained the following theorem [2]:

THEOREM. *A real sequence $\{x_n\}$ is convergent if there exists a regular matrix summability transform T which sums every subsequence of $\{x_n\}$.*

In proving this, the initial observation was made: "We may restrict ourselves to bounded sequences." Since then I have received a number of queries relating to this point, and have privately communicated the proof. Together with several other remarks, this is the substance of this note.

Let $\{x_n\}$ be unbounded, and let T be given by a matrix $[a_{nk}]$. We show that there is a subsequence $\{x_{r_n}\} = \{x'_n\}$ which is not summable T . In discussing the summability of unbounded sequences, the class of regular transforms falls roughly into two subclasses, those (like triangular methods) that apply to *all* unbounded sequences, and those that apply only to unbounded sequences that do not increase too rapidly. In the first case, we shall show that any unbounded sequence x has a subsequence x' whose transform $T(x')$ is unbounded (and certainly divergent); in the second case, we produce a subsequence x' which is not in the domain of T . (Wilansky has informed me that he also has a proof in the case in which T is triangular.)

CASE (i). Let $[a_{n,k}]$ have infinitely many rows which terminate in a string of zeros (have finite length). If T is regular, then the lengths of these cannot be uniformly bounded. Write $a(n, k)$ for $a_{n,k}$, and choose two strictly increasing sequences $\{n_j\}$, $\{k_j\}$ so that $a(n_j, k_j)$ is the last nonzero entry in the n_j th row. Since $x = \{x_n\}$ is unbounded,

Received by the editors January 12, 1956.