SOME MAPPINGS WITH TOPOLOGICAL DEGREE ZERO

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1. **Introduction.** The existence of solutions of an equation

$$M(p) = q$$

is sometimes proved by determining the topological degree of M at q . If the degree is different from zero, the equation has a solution. However if the degree is zero, then purely topological considerations give practically no information about solutions. The advantage of the topological degree approach is that comparatively little information about the mapping is needed. In this note we study locally certain mappings which arise in the study of functional equations and whose topological degree can be computed. By taking advantage of working in the small, we obtain sufficient conditions for the existence or non-existence of solutions if the topological degree of the mapping is zero. Verifying that these conditions hold requires little information beyond that needed to compute the topological degree.

The results we obtain can be applied to obtain existence and non-existence theorems for the nonlinear integral equations and nonlinear elliptic differential equations studied in [3] and [4].

We study mappings in Euclidean 2-space. Our methods apply to similar mappings in n -space ($n > 2$), but if $n > 2$ the problem of computing the topological degree or verifying the conditions derived here is hampered by algebraic complications.

2. **Lemmas.** Consider two mappings, f and g , in the Euclidean plane, defined respectively by the pairs of equations:

$$x'_i = F_i(x_1, x_2), \quad i = 1, 2,$$

and

$$x'_i = G_i(x_1, x_2), \quad i = 1, 2,$$

where F_1, F_2, G_1 , and G_2 and their first derivatives are defined and are continuous on an open set containing a closed circle S of radius r and center at the origin. We denote the origin by θ , and we assume that

$$f^{-1}(\theta) = \theta.$$

A critical point of mapping f is a point in $f(S)$ which is the image

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under f of a point at which the Jacobian of F_1 and F_2 is zero. We suppose that θ is a critical point of f .

LEMMA 1. Suppose the mapping

$$x'_i = - \{G_i(x_1, x_2) - F_i(x_1, x_2)\}, \quad i = 1, 2,$$

takes S into a subsector¹ σ of S , and suppose there is an open set $\mathfrak{U}$ satisfying the following conditions:

- (1) $\mathfrak{U}$ is connected and simply connected;
- (2) there are no critical points of f in $\mathfrak{U}$;
- (3) $\mathfrak{U} \cup [\theta] \supset \sigma$;
- (4) $f(S) \supset \mathfrak{U}$.

Then $\theta \in g(S)$.

LEMMA 2. Suppose there is a set σ_1 of the form

$$\sigma_1 = \{(\rho, \phi) \mid 0 < \rho \leq r, \phi_3 < \phi < \phi_4\}$$

such that $\sigma_1 \cap f(S) = \emptyset$. Suppose the mapping

$$x'_i = - \{G_i(x_1, x_2) - F_i(x_1, x_2)\}, \quad i = 1, 2.$$

takes S into σ_1 . Then $\theta \in g(S)$.

Lemma 2 is trivial. In order to prove Lemma 1, we let C be a component of $f^{-1}(\mathfrak{U})$. Since $\mathfrak{U}$ is open and f is continuous, set C is open. Since there are no critical points of f in $\mathfrak{U}$, it follows that f/C takes open sets into open sets. Because $\mathfrak{U}$ is connected, we have: $f(C) \supset \mathfrak{U}$. Since $\mathfrak{U}$ is simply connected and C is arcwise connected [1, Satz XX, p. 50], it follows by known arguments (see, e.g., [2, proof of Theorem 4.3]) that f is one-to-one on C . Hence f/C is a homeomorphism.

It follows directly that $h = f/C \cup [\theta]$ is also a homeomorphism. We write the continuous mapping h^{-1}/σ as:

$$x_i = \psi_i(x'_1, x'_2), \quad i = 1, 2,$$

Now consider the mapping

$$\begin{aligned} x''_i = & - \{G_i[\psi_i(x'_1, x'_2), \psi_2(x'_1, x'_2)] \\ & - F_i[\psi_i(x'_1, x'_2), \psi_2(x'_1, x'_2)]\}, \quad i = 1, 2, \end{aligned}$$

which acts on σ . This is a continuous mapping of σ into itself and hence by the Brouwer fixed point theorem, has a fixed point (x'_{10}, x'_{20}) . The point

¹ By a sector of S , we mean a set of points $\{(\rho, \phi) \mid 0 \leq \rho \leq r, \phi_1 \leq \phi \leq \phi_2\}$ where ϕ_1, ϕ_2 are given values and $\phi_2, \phi_1, \phi_2 - \phi_1 < 2\pi$. If r is replaced by $r_1 \leq r$, we call the resulting set a subsector of S .

$$x_{i0} = \psi_i(x'_{10}, x'_{20}), \quad i = 1, 2,$$

is mapped into θ by g .

3. Mappings of degree zero. We apply the lemmas to mappings:

$$L_\epsilon: (x_1, x_2) \rightarrow (x'_1, x'_2)$$

defined by the equations:

$$x'_i = P_{m_i}(x_1, x_2) + \tilde{P}_{m_i}(x_1, x_2) + H_i(x_1, x_2, \epsilon), \quad i = 1, 2, \quad |\epsilon| < \epsilon_0,$$

where $P_{m_i}(x_1, x_2)$ is a polynomial homogeneous in x_1 and x_2 of degree $m_i \geq 2$, and $\tilde{P}_{m_i}(x_1, x_2)$ is a term of order exceeding m_i , i.e.,

$$\lim_{r \rightarrow 0} \frac{\tilde{P}_{m_i}(x_1, x_2)}{r^{m_i}} = 0, \quad (r^2 = x_1^2 + x_2^2),$$

and $\tilde{P}_{m_i}(x_1, x_2)$ has continuous first derivatives. The function $H_i(x_1, x_2, \epsilon)$ is a continuous function of (x_1, x_2, ϵ) in a neighborhood of $(0, 0, 0)$, and we assume that $H_i(x_1, x_2, \epsilon)$ is of order n_i (≥ 1) in ϵ in the following sense:

$$H_i(x_1, x_2, \epsilon) = S_i(x_1, x_2)\epsilon^{n_i} + R_i(x_1, x_2, \epsilon)$$

where $S_i(x_1, x_2)$ is a continuous function of (x_1, x_2) ; $\lim_{r \rightarrow 0} S_i(x_1, x_2) = K_i$, a nonzero constant; and

$$\lim_{\epsilon \rightarrow 0} \frac{R_i(x_1, x_2, \epsilon)}{\epsilon^{n_i}} = 0.$$

We assume that the topological index at θ [1, p. 470] of the mapping L given by the equations:

$$x'_i = P_{m_i}(x_1, x_2), \quad i = 1, 2,$$

is defined. Then if ϵ is sufficiently small, the topological degree of L_ϵ at θ relative to a sufficiently small circle with center θ is, by Rouché's Theorem [1, p. 459], equal to the index of L .

Now we use an approach similar to that in [5, §3] to determine when Lemmas 1 and 2 can be applied. Since the methods used are always about the same, we indicate just two applications. First we obtain the real linear factors of odd order of P_{m_1} and P_{m_2} and write:

$$P_{m_1}(x_1, x_2) = p_1(x_1, x_2) \left[\prod_{j=1}^r (x_2 - a_{1j}x_1) \right] a_{10}x_1,$$

$$P_{m_2}(x_1, x_2) = p_2(x_1, x_2) \left[\prod_{k=1}^s (x_2 - a_{2k}x_1) \right],$$

(since the topological index is defined, the factor x_1 can appear in at most one of the two polynomials P_{m_1} and P_{m_2}) where $p_i(x_1, x_2)$ is the product of the complex linear factors and the real linear factors with even exponent of $P_{m_i}(x_1, x_2)$. Since the $p_i(x_1, x_2)$ never change sign, they do not contribute to the topological index except to determine its sign. The number a_{10} may be chosen so that it is less in magnitude than all the numbers $a_{11}, \dots, a_{1r}, a_{21}, \dots, a_{2s}$. Now arrange the numbers $a_{10}, a_{11}, \dots, a_{1r}, a_{21}, \dots, a_{2s}$ in order of magnitude and label them by second subscript accordingly. This tells us the order in which $P_{m_1}(x_1, x_2)$ and $P_{m_2}(x_1, x_2)$ change sign as one proceeds in a counterclockwise direction along a circle with center at θ starting from the intersection of the circle with the negative x_2 -axis. Consequently we see at once the value of the topological index except for sign. If the index is different from zero, then the system of equations

$$L_\epsilon(x_1, x_2) = 0$$

has a solution. We apply Lemmas 1 and 2 to determine when the system has a solution if the index is zero.

It is easy to see that if there is in the ordered array of numbers $a_{10}, a_{11}, \dots, a_{1r}, a_{21}, \dots, a_{2s}$ a subset of the form:

$$(1) \quad a_{1u} < a_{2v} < a_{1,u+1} < a_{2,v+1} < a_{1,u+2}$$

then the set $L(S)$, where S is an arbitrary circle with center at θ , contains a neighborhood of θ . Similarly if such a subset is contained in the array formed if we start from a point on the circle different from its intersection with the negative x_2 -axis (such arrays are obtained from the first array by cyclic permutations), the set $L(S)$ contains a neighborhood of θ . Then the set $L_0(S)$ also contains a neighborhood of θ . This last follows from the higher order properties of the $\tilde{P}_{m_i}$, i.e., we apply the fact that if r is sufficiently small, then

$$|\tilde{P}_{m_i}| < A_i r^{m_i}$$

where A_i is an arbitrarily small positive constant.

The critical points of L are the images of the finite set of lines through θ on which the Jacobian of P_{m_1} and P_{m_2} vanishes. (If the Jacobian vanishes identically, then either the topological index of L is not defined or P_{m_1} and P_{m_2} have no real linear factors.)

THEOREM 1. *If there is a subset of form (1) in the array and if for sufficiently small positive ϵ , the points*

$$(-K_1\epsilon^{n_1}, -K_2\epsilon^{n_2})$$

are not critical points of L , then corresponding to any sufficiently small

circle S with center at θ , there is an $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, then $\theta \in L_\epsilon(S)$.

PROOF. From the continuity and higher order properties of the $\tilde{P}_{m_i}$, it follows that there are positive numbers r_1 and $\epsilon(r_1)$ such that if S_1 , a circle with center θ , is of radius r_1 , there is a sector σ of S_1 such that L_0 has no critical points in an open set

$$\mathfrak{U}_0 = \{(\rho, \phi) / 0 < \rho < r_0, \phi_5 < \phi < \phi_6\}, \quad (r_0 > r_1)$$

which contains $\sigma - \theta$. Also for $0 < \epsilon < \epsilon_1(r_1)$, the points

$$(-K_1\epsilon^{n_1}, -K_2\epsilon^{n_2})$$

are in the interior of σ . There is a circle S_2 with center θ of maximum radius r_2 such that $S_2 \subset L_0(S_1)$, and $\epsilon_1(r_1)$ can be chosen so that if $0 < \epsilon < \epsilon_1(r_1)$, the points

$$(-K_1\epsilon^{n_1}, -K\epsilon^{n_2})$$

are in the interior of $\sigma \cap S_2$. Now positive constants $B_1(r)$, $B_2(r_2)$, and $\epsilon_2 < \epsilon_1(r_1)$ may be chosen such that if $|\xi_1| < B_1(r_2)$, $|\xi_2| < B_2(r_2)$, $0 < \epsilon < \epsilon_2$, the points

$$(-(K_1 + \xi_1)\epsilon^{n_1}, -(K_2 + \xi_2)\epsilon^{n_2})$$

are in the interior of $\sigma \cap S_2$.

Hence there is a circle S_3 with center θ and radius $r_3 < r_1$ and an $\epsilon_3 \leq \epsilon_2$ such that if $(x_1, x_2) \in S_3$ and $\epsilon < \epsilon_3$, then

$$(-H_1(x_1, x_2, \epsilon), -H_2(x_1, x_2, \epsilon)) = (-(K_1 + \xi_1)\epsilon^{n_1}, -(K_2 + \xi_2)\epsilon^{n_2})$$

is in $\sigma \cap S_2$. Choose circles S_4 , S_5 , both with center θ , such that $S_5 \not\subset S_4 \subset L_0(S_3)$. Choose ϵ_4 so that for $0 < \epsilon < \epsilon_4$ and $(x_1, x_2) \in S_3$,

$$(-(K_1 + \xi_1)\epsilon^{n_1}, -(K_2 + \xi_2)\epsilon^{n_2}) \in \sigma \cap S_5.$$

Now apply Lemma 1 letting the mappings f and g be L_0 and L_ϵ where $0 < \epsilon < \epsilon_4$ and letting S be the circle S_3 above and the open set $\mathfrak{U}$ be $\mathfrak{U}_0 \cap (\text{Interior } S_4)$.

The theorem can be sharpened if we investigate the real linear factors of odd order of the Jacobian $J(P_{m_1}, P_{m_2})$ of P_{m_1} and P_{m_2} . We write:

$$J(P_{m_1}, P_{m_2}) = q(x_1, x_2) \left[\prod_{x=l}^t (x_2 - a_{3l}x_1) \right] a_{30}x_1$$

where $q(x_1, x_2)$ is the product of the complex linear factors and the real linear factors with even exponent. (If the factor x_1 appears, we may

choose a_{30} equal to a_{10} and so that it is less in magnitude than all the numbers $a_{11}, \dots, a_{1r}, a_{21}, \dots, a_{2s}, a_{31}, \dots, a_{3t}$.)

We arrange the numbers $a_{10}, a_{11}, \dots, a_{1r}, a_{21}, \dots, a_{2s}, a_{30}, a_{31}, \dots, a_{3t}$ in order of magnitude and label them by second subscript accordingly. This array shows the order in which P_{m_1} and P_{m_2} change sign and also shows the order of the points at which the image of the circle "turns around," i.e., the points where the Jacobian changes sign. Now suppose there is in this array a subset of the form:

$$(2) \quad a_{3w} < a_{1u} < a_{2v} < a_{1,u+1} < a_{2,v+1} < a_{3,w+1}.$$

Let λ_0 and λ_1 be the half-rays in the region $x_1 > 0$ which are contained in the lines:

$$x_2 - a_{3w}x_1 = 0$$

and

$$x_2 - a_{3,w+1}x_1 = 0$$

respectively. Assume that the slope of the tangent at the origin of the curve $L(\lambda_0)$ is less than the slope of the tangent at the origin of the curve $L(\lambda_1)$. Then the set $L(S)$, where S is an arbitrary circle with center at θ , contains a neighborhood of θ and consequently $L_0(S)$ also contains a neighborhood of θ . Thus we may replace in Theorem 1 the hypothesis that there is a subset of the form (1) with the hypothesis that there is a subset of form (2). (As before, we include in this hypothesis the possibility that such a subset is contained in the array obtained if we start from a point on the circle different from its intersection with the negative x_2 -axis.)

Now if P_{m_1} or P_{m_2} has no real linear factors, there is a quadrant Q of the x_1x_2 -plane which does not intersect $L(S)$ except at the origin. Also, returning to the array of numbers $a_{10}, a_{11}, \dots, a_{1r}, a_{21}, \dots, a_{2s}$, if we assume that the array has the form:

$$(3) \quad a_{10} < a_{11} < a_{21} < a_{22} < a_{12} < a_{13} < \dots,$$

i.e., roots of the same polynomial occur in pairs, there is again a quadrant Q which does not intersect $L(S)$ except at the origin.

THEOREM 2. *If one of the polynomials P_{m_1}, P_{m_2} has no real linear factors or if the array of roots is of the form (3) and if for sufficiently small positive ϵ , the points*

$$(-K_1\epsilon^{n_1}, -K_2\epsilon^{n_2})$$

are in the interior of Q , then corresponding to any sufficiently small

circle S with center θ , there is an $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, then $\theta \notin L_\epsilon(S)$.

PROOF. This is an application of Lemma 2. The same type of argument as in the proof of Theorem 1 is applied.

If $\tilde{P}_{m_i}(x_1, x_2)$ is a sum or infinite series of homogeneous polynomials, the methods used here may also be applied in the more complicated case where the topological index of L is not defined but the topological index at θ of the mapping

$$x'_i = P_{m_i}(x_1, x_2) + \tilde{P}_{m_i}(x_1, x_2), \quad i = 1, 2,$$

is defined. We must then consider instead of mapping L , a mapping

$$x'_i = \sum_{q=0}^{r_i} P_{m_i+q}(x_1, x_2), \quad i = 1, 2,$$

where the $P_{m_i+q}(x_1, x_2)$ are homogeneous polynomials of degree m_i+q . Some results can be obtained, but the computations are so much more complicated that it seems practical to work directly with each given mapping rather than attempt to obtain general theorems.

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