ON THE FINITENESS OF FREE (UNIVERSAL) ALGEBRAS

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1. Introduction. In the present note we exhibit a generalization—to universal algebras—of the familiar structure of free Boolean algebras, particularly the free algebras of a finite number of generators. The result here established is a ready consequence of the principal structure theorem for (universal) “ $\mathfrak{P}$ -algebras” (Foster [1; 2]; see also below); from these references we recall several orienting essentials.

Let $S\mathfrak{p} = (n_1, n_2, \dots)$ be a given species and $\mathfrak{A} = (A, o_1, o_2, \dots)$ a (universal) algebra of species $S\mathfrak{p}$; here $A = \{\dots, \xi, \dots\}$ is the class of elements of $\mathfrak{A}$ and $o_1, o_2, \dots$ the primitive operations, where o_i is n_i -ary,

$$o_i(\xi_1, \dots, \xi_{n_i}) \in A, \quad (\xi_1, \dots, \xi_{n_i} \in A; i = 1, 2, \dots).$$

$\mathfrak{A}$ is called “functionally strictly complete” (f.s.c.) if for each positive integer k and for each mapping $f(\xi_1, \dots, \xi_k)$ of the set $A^{(k)}$ ($=k$ th Cartesian power of A) into the set A , the function $f(\xi_1, \dots, \xi_k)$ is expressible as a “strict $\mathfrak{A}$ -function,” i.e., as some composition, via the primitive o_i of $\mathfrak{A}$, of indeterminates $\xi_1, \dots, \xi_k$ over the set A . If $\mathfrak{A}$ is f.s.c. it is necessarily of finite order n (A = class of n elements); also for at least one n_i of $S\mathfrak{p}$, $n_i > 1$ (see [1; 2]).

A “primal” (universal) algebra $\mathfrak{P} = (P, o_1, o_2, \dots)$ —of species $S\mathfrak{p}$ —is one which is f.s.c. and not isomorphic with 1° , the one-element algebra. Among the profusion of known primal algebras we here recall (1°) $(F_2, \times, \neg)$, the two element Boolean algebra, and more generally (2°) $(F_p, \times, \neg)$, where $\times, \neg$ are respectively the product and $\xi^\neg = 1 + \xi$ in $(F_p, \times, +)$, the prime field of characteristic p . Again, for any integer $n > 1$, (3°) $(P_n, \times, \neg)$, the basic Post algebra of order n (Rosenbloom [4]; Foster [1; 2]). Further we have (4°) , $(G_n, \times, \neg)$, any “group frame”: here $(G_n, \times)$ is $(G', \times)$ —any finite group of order $n - 1$ and with its identity denoted by 1 —augmented by a null element, 0 , where $0 \times \xi = \xi \times 0 = 0$ ($\xi \in G$), and $\xi^\neg$ is defined as the cyclic permutation of G corresponding to any given ordering $0, 1, \alpha, \beta, \dots, \gamma$ of G , i.e., $0^\neg = 1, 1^\neg = \alpha, \alpha^\neg = \beta, \dots, \gamma^\neg = 0$. (For these and additional primal algebras see [1; 2].)

Each primal algebra $\mathfrak{P}$ is the “kernel” of the class of “ $\mathfrak{P}$ -algebras,” that is, the class of algebras $\mathfrak{A}$, of the same species as $\mathfrak{P}$, which

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satisfy all identities satisfied by $\mathfrak{P}$. In particular the class of all (i) Boolean algebras, (ii) p -algebras (rings), (iii) Post algebras, is coextensive, up to isomorphisms, with the class of all $\mathfrak{P}$ -algebras with the kernel $\mathfrak{P}$ respectively chosen as (i) $(F_2, \times, \neg)$, (ii) $(F_p, \times, \neg)$, (iii) $(P_n, \times, \neg)$.

From [2] we require the following basic result.

THEOREM 1. *Let $\mathfrak{P}$ be any primal algebra. Then: (1) The class of all $\mathfrak{P}$ -algebras is coextensive with the class of all subdirect powers of $\mathfrak{P}$, or, equivalently, with the class of all subalgebras of direct powers of $\mathfrak{P}$. (2) A finite $\mathfrak{P}$ -algebra, $\mathfrak{A}$, is isomorphic with a direct power of $\mathfrak{P}$,*

$$\mathfrak{A} \cong \mathfrak{P}^{(t)} = \mathfrak{P} \times \mathfrak{P} \times \cdots (= t\text{th direct power}),$$

and

$$n' = n^t \quad (n', n = \text{orders of } \mathfrak{A}, \mathfrak{P}).$$

2. Free algebras. Let $\mathfrak{S}$ be a class of equations, of species Sp . A model of $\mathfrak{S}$, i.e., an algebra $\mathfrak{A}$ of species Sp , which satisfies all identities $\mathfrak{S}$, is called "exact" if each identity satisfied by $\mathfrak{A}$ is a logical consequence of $\mathfrak{S}$.

THEOREM 2. *Let $\mathfrak{S}$ be a class of equations of species Sp , let k be an integer and let $\mathfrak{F}_k(\mathfrak{S})$ be the free algebra of k generators determined by $\mathfrak{S}$. If $\mathfrak{S}$ possesses a (finite) model, $\mathfrak{P}$, which is (1°) exact and (2°) functionally strictly complete, then $\mathfrak{F}_k(\mathfrak{S})$ is a finite algebra with the structure*

$$\mathfrak{F}_k(\mathfrak{S}) \cong \mathfrak{P}^{(n^k)} (= n^k\text{th direct power of } \mathfrak{P}; n = \text{order of } \mathfrak{P}),$$

$$n' = n^{n^k} \quad (n' = \text{order of } \mathfrak{F}_k(\mathfrak{S})).$$

PROOF. Let $\xi_1, \dots, \xi_k$ be generators of $\mathfrak{F}_k(\mathfrak{S})$. Since $\mathfrak{P}$ is assumed to be an exact model of $\mathfrak{S}$, the elements of $\mathfrak{F}_k(\mathfrak{S})$ are in 1-1 correspondence with the classes $C_1, C_2, \dots$ of identical strict $\mathfrak{P}$ -functions $f(\xi_1, \dots, \xi_k)$, i.e., $f(\xi_1, \dots, \xi_k)$ and $g(\xi_1, \dots, \xi_k)$ belong to the same C_i if and only if $f=g$ is an identity of $\mathfrak{P}$. Now since $\mathfrak{P} = (P, o, \dots)$ is further assumed to be functionally strictly complete, each of the n^{n^k} different set-theoretical mappings $f(\xi_1, \dots, \xi_k)$ of the set $P^{(k)}$ ($=k$ th Cartesian power of P) into the set P may be expressed in at least one way as a strict $\mathfrak{P}$ -function $f(\xi_1, \dots, \xi_k)$. Hence there are precisely n^{n^k} classes C_i and consequently

$$(1) \quad n' = \text{order of } \mathfrak{F}_k(\mathfrak{S}) = n^{n^k}.$$

In particular $\mathfrak{F}_k(\mathfrak{S})$ is thus a finite $\mathfrak{P}$ -algebra. From (2) of Theorem 1 we then have: for some integer t ,

$$(2) \quad \mathfrak{F}_k(\mathfrak{S}) \cong \mathfrak{P}^{(t)} (= t\text{th direct power of } \mathfrak{P}).$$

Comparing the order of $\mathfrak{F}_k(\mathfrak{S})$ computed from (2) with (1) we have $t = n^k$ and the theorem is proved.

REMARK. The finiteness of $\mathfrak{F}_k(\mathfrak{S})$ —apart from its structure—follows from condition (1°) alone. In fact if $\mathfrak{S}$ possesses a finite and exact model, $\mathfrak{A}$, of order n , the number of classes C_i above is $\leq n^n$, and we have

$$n' = \text{order of } \mathfrak{F}_k(\mathfrak{S}) \leq n^{n^k}.$$

If in Theorem 2 we take $\mathfrak{S}$ as the class $\mathcal{S}(\mathfrak{P})$ of all strict identities satisfied by $\mathfrak{P}$ or, what is equivalent, as any basis $I_1, I_2, \dots$ for the class $\mathcal{S}(\mathfrak{P})$, Theorem 2 may be restated in the form of Theorem 3 below. (In this connection it is known (Yaqub [5]) that $\mathcal{S}(\mathfrak{P})$ always possesses a *finite* basis, i.e., a finite number of strict identities $I_1, I_2, \dots, I_s$ satisfied by $\mathfrak{P}$ from which all strict identities satisfied by $\mathfrak{P}$ follow as a logical consequence.)

THEOREM 3. Let $\mathfrak{P} = (P, o, \dots)$ be an arbitrary primal algebra of order n . Then $\mathfrak{F}_k[\mathfrak{P}]$, the free $\mathfrak{P}$ -algebra of a finite number k of generators, has the structure

$$\mathfrak{F}_k[\mathfrak{P}] \cong \mathfrak{P}^{(n^k)} \quad (= n^{\text{th}} \text{ direct power of } \mathfrak{P}),$$

$$n' = \text{order of } \mathfrak{F}_k[\mathfrak{P}] = n^{n^k}.$$

On proper choice of the kernel, $\mathfrak{P}$, Theorem 3 applies, in particular, to free Boolean algebras, free p -algebras (rings), free Post-algebras, etc. (§1).

Finally we remark that Theorem 3 (or 2) may be further generalized to yield a corresponding structure for $\mathfrak{F}_k[\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_s]$, the free “ $(\mathfrak{P}_1, \dots, \mathfrak{P}_s)$ -algebra.” This extension requires the general structure theorem for such algebras, derived in [3]. Here $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_s$ is any “independent” class (see [3] for definition) of primal algebras (all of the same species Sp), and a “ $(\mathfrak{P}_1, \dots, \mathfrak{P}_s)$ -algebra” is any algebra, $\mathfrak{A}$, (of species Sp) which satisfies all strict identities common to $\mathfrak{P}_1, \dots, \mathfrak{P}_s$. We shall however not enter into this here.

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