ON THE ABSOLUTE CONVERGENCE OF A SERIES ASSOCIATED WITH A FOURIER SERIES

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1. Suppose $f(t)$ is integrable L in $(-\pi, \pi)$ periodic with period 2π, and that its Fourier series at $t=x$ is

$$\frac{1}{2} a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) = \sum A_n \tag{1.1}$$

and its conjugate series at $t=x$ is

$$\sum_1^\infty (b_n \cos nx - a_n \sin nx) = \sum B_n. \tag{1.2}$$

We shall be concerned in this note with the series

$$\sum \frac{s_n - s}{n} \tag{1.3}$$

where

$$S_n = \sum_1^n A_k \tag{1.4}$$

and s is an appropriate number independent of n.

We write

$$\phi(t) = \{f(x+t) + f(x-t) - 2s\}/2, \tag{1.5}$$

$$\phi_\alpha(t) = \alpha t^{-\alpha} \int_0^t (t-u)^{\alpha-1} \phi(u) du, \qquad \alpha > 0, \text{ and } \phi_0(t) = \phi(t), \tag{1.6}$$

$$\psi(t) = \{f(x+t) - f(x-t)\}/2. \tag{1.7}$$

The ordinary Cesàro summability of the series (1.3) was first studied by Hardy and Littlewood [5] who observed that the relation of (1.3) to the integral $\int_0 (\phi(t)/t) dt$ is very similar to that between the allied series and the integral $\int_0 (\psi(t)/t) dt$. Zygmund [10] gave a necessary and sufficient condition for the convergence of the same series (1.3). The object of the present note is to study the absolute convergence and absolute Riesz summability of (1.3). We prove the following

THEOREM 1. *If* (1) $\phi_1(t) \log k/t$ *is of bounded variation in* $(0, \pi)$,

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(2) $|\phi_1(t)|/t$ *is integrable in* $(0, \pi)$, (3) $\{n^\delta A_n\}$ *is of bounded variation*, $\delta > 0$, *then* (1.3) *is absolutely convergent.*

THEOREM 2. *If the series* (1.3) *is absolutely convergent then* $\int_0^\pi (|\phi_{1+\delta}(t)|/t)dt < \infty$, $\delta > 0$.

THEOREM 3. *If* $|\phi(t)|/t$ *is integrable in* $(0, \pi)$ *then the series* (1.3) *is summable* $|C, \delta|$, $\delta > 0$.

In Theorems 1 and 1(a), and the lemmas which follow, k is any number greater than π.

2. For proving Theorem 1 we first prove

THEOREM 1(a). *If* (1) $\phi_1(t) \log (k/t)$ *is of bounded variation in* $(0, \pi)$, (2) $|\phi_1(t)|/t$ *is integrable in* $(0, \pi)$, *then the series* (1.3) *is summable* $|R, e^{n^\alpha}, 1|$, $0 < \alpha < 1$.

We require the following lemmas for the proof of Theorem 1(a).

LEMMA 1. *If* $\phi(t) \log (k/t)$ *is of bounded variation in* $(0, \pi)$, *then the series* (1.1) *is summable* $|R, e^{n^\alpha}, 1|$, $0 < \alpha < 1$.

LEMMA 2. *If* $\psi(t) \log (k/t)$ *is of bounded variation in* $(0, \pi)$ *and* $|\psi(t)|/t$ *is integrable in* $(0, \pi)$, *then the series* (1.2) *is summable* $|R, e^{n^\alpha}, 1|$, $0 < \alpha < 1$.

LEMMA 3. *If* $\phi(t) \log (k/t)$ *is of bounded variation in* $(0, \pi)$ *and* $|\phi(t)|/t$ *is integrable in* $(0, \pi)$, *then the series* (1.3) *is absolutely convergent.*

Lemmas 1 and 2 are known [6], whereas Lemma 3 with relaxed hypothesis can be proved by Tauberian argument, but we give the following direct proof which will suffice for our purpose.

PROOF OF LEMMA 3. Since $\phi(t)$ is of bounded variation in $(0, \pi)$, we have

$$\frac{s_n - s}{n} = \frac{1}{2\pi} \frac{1}{n} \int_0^\pi \phi(t) \frac{\sin nt}{t} dt + O(n^{-2}) = t_n + O(n^{-2}) \text{ say.}$$

Now

$$t_n = \frac{-1}{2\pi} \frac{1}{n} \int_0^\pi d\phi U_n(t);$$

on integration by parts, where $U_n(t) = \int_t^\pi \sin (nu/u)\, du$. It will be enough to prove that $\sum |t_n| < \infty$.

Now,

$$\sum |t_n| = \frac{1}{2\pi}\int_0^\pi |d\phi| \sum \frac{1}{n} |U_n(t)|$$
$$\leqq A\int_0^\pi |d\phi| \left\{ \sum_{n\leqq kt^{-1}} \frac{1}{n} + \frac{1}{t} \sum_{n>kt^{-1}} \frac{1}{n^2}\right\}$$
$$< A\int_0^\pi |d\phi| \log\frac{k}{t} < \infty,$$

using fairly obvious inequalities for $U_n(t)$, viz.

$$|U_n(t)| \leqq \begin{cases} A, \\ A(nt)^{-1}. \end{cases}$$

We now begin to prove Theorem 1(a).

$$\frac{s_n - s}{n} = \frac{1}{n\pi}\int_0^\pi \phi(t) \frac{\sin (n+1/2)t}{2\sin (t/2)} dt$$
$$= \frac{1}{2\pi}\int_0^\pi \phi(t) \cot (t/2) \frac{\sin nt}{n} dt + \frac{1}{2\pi}\int_0^\pi \phi(t) \frac{\cos nt}{n} dt$$
$$= \frac{1}{2\pi}(\quad \alpha_n \quad + \quad \beta_n), \text{ say.}$$

On integration by parts

$$\alpha_n = -\int_0^\pi \frac{\phi_1(t)}{t}\left(\frac{t/2}{\sin (t/2)}\right)^2 \frac{\sin nt}{n} dt$$
$$+ \frac{1}{2}\int_0^\pi \phi_1(t) t \cot (t/2) \cos nt dt$$
$$= -\int_0^\pi \frac{F(t)}{t} \frac{\sin nt}{n} dt + \frac{1}{2}\int_0^\pi G(t) \cos nt\, dt = P_n + Q_n, \text{ say,}$$

where

$$F(t) = \phi_1(t)\left(\frac{t/2}{\sin (t/2)}\right)^2 \quad \text{and} \quad G(t) = t\phi_1(t) \cot (t/2).$$

$\sum|P_n|$ is convergent by Lemma 3 with $F(t)$ in place of $\phi(t)$ and $\sum Q_n$ is summable $|R, e^{n^\alpha}, 1|$, $0<\alpha<1$, by Lemma 1 with $G(t)$ in place of $\phi(t)$ and hence $\sum\alpha_n$ is summable $|R, e^{n^\alpha}, 1|$. On integration by parts,

$$\beta_n = \Phi(\pi) \frac{\cos n\pi}{n} - \frac{1}{2\pi}\int_0^\pi \{\Phi(t)\} \sin ntdt, = \gamma_n + \delta_n, \text{ say.}$$

$\sum \gamma_n$ is summable $|R, e^{n^\alpha}, 1|$ by known results [7] and the discussion of $|R, e^{n^\alpha}, 1|$ summability of $\sum \delta_n$ is similar to that of the conjugate series $\sum B_n$ with $t\phi_1(t)$ taking place of $\psi(t)$ and therefore by Lemma 2 it follows that $\sum \delta_n$ is summable $|R, e^{n^\alpha}, 1|$ if $\int_0^\pi |d\{t\phi_1(t)\}| \log k/t < \infty$, which is satisfied under the hypothesis of Theorem 1(a), since

$$\int_0^\pi |d\{t\phi_1(t)\}| \log \frac{k}{t} \leqq \int_0^\pi |\phi_1(t)| \log \frac{k}{t} dt + \int_0^\pi t|d\phi_1| \log \frac{k}{t}$$

and both the integrals on the right are convergent by the hypothesis of Theorem 1(a) and thus the proof of Theorem 1(a) is complete. Combining Theorem 1(a) with the following known theorem [9], proof of Theorem 1 is completed with $(s_n - s)/n$ in place of a_n.

THEOREM A. *If* (1) $\sum a_n$ *is summable* $|R, e^{n^\alpha}, 1|$, (2) $\{n^{1-\alpha}a_n\}$ *is of bounded variation, then* $\sum |a_n|$ *is convergent.*

Under hypotheses (1) and (2) of Theorem 1 the series (1.3) is summable $|R, e^{n^\alpha}, 1|$ and it would be absolutely convergent, if $\{n^{1-\alpha}(s_n - s)/n\}$ is of bounded variation, i.e. if $\{(s_n - s)/n^\alpha\}$ is of bounded variation, i.e. if $\{s_n/n^\alpha\}$ is of bounded variation.

Now

$$\Delta\left\{\frac{s_n}{n^\alpha}\right\} = \frac{s_n}{n^\alpha} - \frac{s_{n-1}}{(n-1)^\alpha} = \frac{s_n - s_{n-1}}{n^\alpha} + s_{n-1}\left\{\frac{1}{n^\alpha} - \frac{1}{(n-1)^\alpha}\right\}$$
$$= \frac{A_n}{n^\alpha} - s_{n-1}\left\{\frac{1}{(n-1)^\alpha} - \frac{1}{n^\alpha}\right\}.$$

Therefore

$$\sum \left|\Delta\left(\frac{s_n}{n^\alpha}\right)\right| \leqq \sum \frac{|A_n|}{n^\alpha} + \sum |s_{n-1}| \left\{\frac{1}{(n-1)^\alpha} - \frac{1}{n^\alpha}\right\}$$
$$= \sum \frac{|A_n|}{n^\alpha} + O(1)$$

by hypothesis 1 of Theorem 1 which also ensures $|C|$ summability of $\sum A_n$ and a fortiori that of $\sum A_n/n^\alpha$.

Hence $\sum |A_n|/n^\alpha$ will be convergent by the appropriate tauberian theorem above if the sequence $\{n(A_n/n^\alpha)\}$ is of bounded variation, i.e. if $\{n^{1-\alpha}A_n\}$ is of bounded variation, i.e. if $\{n^\delta A_n\}$ is of bounded variation, writing $\alpha = 1 - \delta$, which is the condition (3) of Theorem 1.

3. PROOF OF THEOREM 2. We have [1] $\phi_{1+\delta}(t) = (1+\delta) \sum s_n \gamma_{1+\delta}(nt)$ where $\gamma_\sigma(x) = \int_0^1 (1-u)^{\sigma-1} \cos uxdu$. Now

$$\int_0^\pi \frac{|\phi_{1+\delta}(t)|}{t}\,dt \leqq \sum |s_n| \int_0^\pi |\Delta\gamma_{1+\delta}(nt)| \frac{dt}{t}$$
$$\leqq A \sum \frac{|s_n|}{n} < \infty$$

by using appropriate inequalities for $\Delta\gamma_{1+\delta}(nt)$ [2].

REMARKS. It is worth remarking here that if $f(x)$ belongs to Lip α, $0<\alpha<1$, i.e. $f(x+h)-f(x)=O\{|h|^\alpha\}$ uniformly as $h\to 0$, then $\sum|s_n-f(x)|/n$ converges uniformly in x. For it is well known [10] that when $f(x)$ belongs to Lip α, then $|s_n-f(x)|=O\{\log n/n^\alpha\}$ uniformly in x. But the Fourier series of a function belonging to Lip α is absolutely convergent when $\alpha>1/2$. Again the Fourier series of a function can be absolutely convergent at a point without the series (1.3) being absolutely convergent at that point. For instance, if $\phi(t)=(\log(k/|t|))^{-1}$ in $(0, \pi)$ and defined by periodicity elsewhere, it is known that [8] $A_n=O\{(1/n(\log n)^2)\}$ and hence $\sum|A_n|<\infty$, i.e. the Fourier series of $\phi(t)$ at $t=0$ is absolutely convergent. But $\int_0^\pi|\phi_\lambda(t)|/t$ is not convergent for any λ and hence the series can not be absolutely convergent by Theorem 2 of the present note.

4. The proof of Theorem 3 can be effected in the same manner as that of a known result [3].

From this theorem it is clear that $|C, \delta|$ summability of (1.3) is a local property of its generating function whereas its absolute convergence is not so, [4].

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