ON A TEST FOR THE CONVERGENCE OF FOURIER SERIES

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Suppose that $\phi(t)$ is an even and integrable function, periodic with period 2π . Let

$$\phi(t) \sim \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} a_{\nu} \cos \nu t.$$

Wang¹ has established the following two theorems.

THEOREM A. *If*

$$(i) \quad \int_0^t \phi(u) du = o\left(\frac{t}{\log(1/t)}\right) \quad (t \rightarrow +0),$$

$$(ii) \quad a_n > -K \frac{\log n}{n}$$

for some $K > 0$, then the series $\sum a_n$ converges.

THEOREM B. *There exists an even function $\phi(t)$, satisfying (i), whose Fourier series diverges at $t=0$, while*

$$(ii') \quad a_n = o\left(\frac{(\log n)^2}{n}\right).$$

The problem of bridging the gap between Theorems A and B is still unsolved. In this note, we remove this gap by proving the following

THEOREM. *There exists an even function $\phi(t)$, satisfying (i), whose Fourier series diverges at $t=0$, while*

$$(ii'') \quad a_n = o\left(\frac{(\log n)^{1+\eta}}{n}\right),$$

where η is any preassigned positive value.

PROOF. We choose three sequences of positive values (λ_i) , (α_i) and (c_i) such that

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¹ Proc. London Math. Soc. (2) vol. 47 (1942) p. 308.

$$\begin{aligned} 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \rightarrow \infty;^2 \\ \alpha_i &= \log \lambda_i / \lambda_i & (i \geq 1), \alpha_0 = \pi; \\ c_i &= (\log \alpha_{i-1} / \alpha)^{-1/(1+\eta)}. \end{aligned}$$

And define an even function:

$$\phi(t) = c_i \sin \lambda_i t \quad (\alpha_i < t \leq \alpha_{i-1}; i = 1, 2, 3, \dots).$$

Then, $\phi(t)$, being bounded, is integrable in the Lebesgue sense.

By a number of direct calculations,³ we can arrive at the following results:

$$(1) \quad \frac{1}{t} \log \frac{1}{t} \int_0^t \phi(u) du = \frac{1}{\alpha_n} \log \frac{1}{\alpha_n} O\left(\sum_{i \geq n} \frac{c_i}{\lambda_i}\right) \text{ for } \alpha_n < t \leq \alpha_{n-1}.$$

Let the λ_n th partial sum of the Fourier series of $\phi(t)$ at $t=0$ be S_{λ_n} , then

$$\begin{aligned} S_{\lambda_n} &= \frac{2}{\pi} \int_0^\pi \phi(t) \frac{\sin \lambda_n t}{t} dt + o(1) \\ &= \frac{2}{\pi} \left(\sum_{i < n} + \sum_{i > n} \right) c_i \int_{\alpha_i}^{\alpha_{i-1}} \frac{\sin \lambda_i t \sin \lambda_n t}{t} dt \\ &\quad + \frac{2c_n}{\pi} \int_{\alpha_n}^{\alpha_{n-1}} \frac{\sin^2 \lambda_n t}{t} dt + o(1) \\ &= \frac{2}{\pi} (\Sigma_1 + \Sigma_2 + \Sigma_3) + o(1). \end{aligned}$$

We have

$$\begin{aligned} (2) \quad \Sigma_1 &= \sum_{i < n} c_i \int_{\alpha_i}^{\alpha_{i-1}} \frac{\sin \lambda_i t \sin \lambda_n t}{t} dt \\ &= O\left(\sum_{i < n} \frac{c_i}{\alpha_i(\lambda_n - \lambda_i)}\right) = \frac{1}{\lambda_n} O\left(\sum_{i < n} \frac{c_i}{\alpha_i}\right), \end{aligned}$$

if $\lambda_{n-1}/\lambda_n < k < 1$ for every n .

$$(3) \quad \Sigma_2 = \sum_{i > n} c_i \int_{\alpha_i}^{\alpha_{i-1}} \frac{\sin \lambda_i t \sin \lambda_n t}{t} dt = \lambda_n O\left(\sum_{i > n} \frac{c_i \alpha_{i-1}}{\lambda_i \alpha_i}\right),$$

² We take the sequence (λ_i) in such a manner that $\alpha_i < \alpha_{i-1}$ for every i and $c_i = o(1)$ as $i \rightarrow \infty$.

³ For the detailed derivation of these expressions, cf. Proc. London Math. Soc. loc. cit., pp. 316-318.

since the total variation of $\sin \lambda_n t$ in (α_i, α_{i-1}) is $O(\alpha_{i-1} \lambda_n)$.

$$(4) \quad \Sigma_3 = c_n \int_{\alpha_n}^{\alpha_{n-1}} \frac{\sin^2 \lambda_n t}{t} dt = \frac{c_n}{2} \log \frac{\alpha_{n-1}}{\alpha_n} + O\left(\frac{c_n}{\alpha_n \lambda_n}\right).$$

Finally, we have

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi \phi(t) \cos ntdt = \frac{2}{\pi} \left(\sum_{i \leq \rho_n} + \sum_{i > \rho_n} \right) c_i \int_{\alpha_i}^{\alpha_{i-1}} \sin \lambda_i t \cos ntdt \\ &= \frac{2}{\pi} \left(\sum_{i \leq \rho_n} J_i + \sum_{i > \rho_n} J_i \right) \end{aligned}$$

say, where (ρ_n) is a rapidly increasing sequence of positive integers.

Now,

$$(5) \quad \sum_{i \leq \rho_n} J_i = \sum_{i \leq \rho_n} c_i \int_{\alpha_i}^{\alpha_{i-1}} \sin \lambda_i t \cos ntdt = \frac{1}{n} O\left(\sum_{i \leq \rho_n} c_i \lambda_i \alpha_{i-1}\right).$$

On the other hand,

$$(6) \quad \sum_{i > \rho_n} J_i = O\left(\sum_{i > \rho_n} \frac{c_i}{|\lambda_i - n|}\right).$$

In order to establish the theorem, we need to make the value in (1), $o(1)$, $\Sigma_1 + \Sigma_2 = o(1)$, $\Sigma_3 > 1$ and the values of the expressions in (5) and (6) are both $o((\log n)^{1+\eta}/n)$.

We take $\lambda_i = [(i+1)!]^\eta$. Then

$$\frac{1}{t} \log \frac{1}{t} \int_0^t \phi(u) du = ((n+1)!)^\eta o\left(\sum_{i \geq n} \frac{1}{((i+1)!)^\eta}\right) = o(1)$$

as $t \rightarrow +\infty$. And

$$\Sigma_1 = \frac{1}{((n+1)!)^\eta} O\left(\sum_{i < n} \frac{((i+1)!)^\eta}{\log(i+1)!}\right)$$

$$= \frac{1}{((n+1)!)^\eta} O\left(\frac{(n!)^\eta}{\log n}\right) = o(1).$$

$$\Sigma_2 = ((n+1)!)^\eta o\left(\sum_{i > n} \frac{1}{(i!)^\eta}\right) = o(1).$$

$$\Sigma_3 \cong \frac{\eta}{2(\log(n+1))^{1/(1+\eta)}} \log \frac{(n+1)!}{n!} + o(1)$$

$$= \frac{\eta}{2} (\log(n+1))^{\eta/(1+\eta)} + o(1) > 1.$$

This shows that the Fourier series of $\phi(t)$ diverges at $t=0$.

Moreover,

$$\sum_{i \leq \rho_n} J_i = \frac{1}{n} o \left(\sum_{i \leq \rho_n} ((i+1)!)^\eta \frac{\log i!}{(i!)^\eta} \right) = \frac{1}{n} o(\rho_n^{2+\eta} \log \rho_n),$$

$$\sum_{i > \rho_n} J_i = o \left(\sum_{i > \rho_n} \frac{1}{((i+1)!)^\eta - n} \right) = o \left(\sum_{i > \rho_n} \frac{1}{((i+1)!)^\eta - (i!)^\eta} \right),$$

since $(i!)^\eta \geq \lambda_{i-1} \geq \lambda_{\rho_n} > n$. Therefore,

$$\sum_{i > \rho_n} J_i = o \left(\frac{1}{(\rho_n!)^\eta} \right) = o \left(\frac{e^{\eta \rho_n}}{\rho_n^{\eta(\rho_n+1/2)}} \right)$$

by Stirling's formula

$$e^n n! \cong (2\pi)^{1/2} n^{n+1/2}.$$

Now, if we choose a rapidly increasing sequence of positive integers (n_i) such as $n_i = ((i+1)!)!$ and define another function $\phi^*(t)$ from $\phi(t)$ as follows:

$$\phi^*(t) = c_i^* \sin \lambda_i^* t \quad (\alpha_i^* < t \leq \alpha_{i+1}^*),$$

where $\lambda_i^* = \lambda_i$, $\alpha_i^* = \alpha_i$ and $c_i^* = c_{n_i}$ if $v = n_i$ for a certain i ; and $c_i^* = 0$ if $v \neq n_i$ for every i , under this circumstance, the corresponding condition (i) of the theorem is still satisfied and the Fourier series of $\phi^*(t)$ still diverges at $t=0$ as before. But, by this method of choice of the sequence (n_i) , we can easily find that the corresponding

$$\sum_{i \leq \rho_n} J_i^* = \frac{1}{n} o(\rho_n^{1+\eta} \log \rho_n).$$

Taking $\rho_n = [\log n (\log^2 n)^{-1/(1+\eta)}]$, we obtain

$$\sum_{i \leq \rho_n} J_i^* = o \left(\frac{(\log n)^{1+\eta}}{n} \right),$$

also,

$$\sum_{i > \rho_n} J_i^* = o \left(\frac{(\log n)^{1+\eta}}{n} \right).$$

Thus, $a_n^* = o((\log n)^{1+\eta}/n)$. This completes the proof of the theorem.