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# A NOTE ON COMPLETELY PRIMARY RINGS

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A completely primary ring will mean a commutative ring with identity in which the ideal of nilpotent elements, called the radical, is a maximal ideal. For a completely primary ring $A$ with radical $N$, $\bar{A}$ will mean the quotient ring $A/N$. It has been shown by E. Snapper[1] that if $A$ is a completely primary ring of characteristic zero then $A$ contains a field $F$ isomorphic with $\bar{A}$. The purpose of this note is to extend and, incidentally, give another proof of Snapper's result.

Theorem. *If $A$ is a completely primary ring of characteristic zero and $N$ its radical, then $A$ contains a field $F$ isomorphic with $\bar{A}=A/N$ such that each $a$ in $A$ can be uniquely written in the form $a=f+n$, where $f\in F$, $n\in N$.*

Proof. First note that $x\notin N$ implies that $x$ has an inverse, $x^{-1}$. By Zorn's lemma $A$ contains a maximal ring $F$ whose intersection with $N$ is 0. This ring $F$ is a field, for otherwise the set $F^*$ of all elements of the form $ab^{-1}$, $0\neq b$, $a\in F$, is a field containing $F$, whose intersection with $N$ is 0, a contradiction. To prove the theorem it is sufficient to show that $A$ is identical with the subset $A^*$ of $A$ consisting of all

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[1] E. Snapper, *Completely primary rings*, III *Imbedding and isomorphism theorems*, Ann. of Math. vol. 53 (1951) pp. 207–234. See Remark 4.1, p. 218.

elements of the form $f+n$, $f\in F$, $n\in N$. Assume the contrary, then there is an $x\in A$ such that $x\notin A^*$. For $z\in A$ denote by $\bar{z}$ the image in $\bar{A}$ of $z$ under the natural homomorphism of $A$ onto $\bar{A}$, and by $\bar{F}$ the set of all $\bar{f}$, $f\in F$. Note that $\bar{F}$ is a subfield of $\bar{A}$ isomorphic with $F$. Since $x\notin A^*$, $\bar{x}\neq\bar{0}$ and $\bar{x}\notin\bar{F}$. Suppose that $\bar{x}$ is transcendental over $\bar{F}$. Then for any element $g=f_0+f_1x+\cdots+f_nx^n$, in the subring $F[x]$ of $A$ generated by $F$ and $x$, it is clear that $g\in N$ implies $g=0$, or $N\cap F[x]=0$. This, however, contradicts the maximality of $F$, since $F[x]\supset F$. Thus, $\bar{x}$ must be algebraic over $\bar{F}$.

Now, let $\bar{f}(\lambda)=\bar{f}_0+\bar{f}_1\lambda+\cdots+\bar{f}_m\lambda^m$ be a minimum polynomial for $\bar{x}$ over $\bar{F}$. Denote by $n(r)$ the element $f_0+f_1(x+r)+\cdots+f_m(x+r)^m$ in the subring $F_r$ of $A$ generated by $F$ and $x+r$, and note that for each $r\in N$, $n(r)\in N$. Further, if $h=h_0+h_1(x+r)+\cdots+h_t(x+r)^t\in F_r$, with $t<m$, then $h\notin N$. It will be shown that for some $r\in N$, $n(r)=0$. Assume the contrary and choose $r_0$ so that $n(r_0)$ has minimum exponent $\rho>1$. Let $x_1=x+r_0$ and $f^{(i)}(x_1)=a_0+a_1x_1+\cdots+a_{m-i}x_1^{m-i}$, where $a_j=(i+j)(i+j-1)\cdots(j+1)f_{i+j}\in F$. In particular, $f'(x_1)=f_1+2f_2x_1+\cdots+mf_mx_1^{m-1}\notin N$, and hence has an inverse in $A$. Let $r_1=-[f'(x_1)]^{-1}n(r_0)$, so that

$$
\begin{aligned}
n(r_0+r_1) &= f_0+f_1(x+r_0+r_1)+\cdots+f_m(x+r_0+r_1)^m \\
&= n(r_0)+f'(x_1)r_1+\frac{1}{2!}f^{(2)}(x_1)r_1^2+\cdots+\frac{1}{m!}f^{(m)}(x_1)r_1^m \\
&= \left[\frac{1}{2!}f^{(2)}(x_1)+\cdots+\frac{1}{m!}f^{(m)}(x_1)r_1^{m-2}\right]r_1^2 = c[n(r_0)]^2,
\end{aligned}
$$

has exponent less than $\rho$. This is a contradiction and hence $n(r_0)=0$.

Consider next the subring $F_{r_0}$ of $A$. Recall that this is the subring generated by $F$ and $x_1=x+r_0$. Since $f(x_1)=n(r_0)=0$ it is clear that the kernel of the homomorphism, $g(\lambda)\rightarrow g(x_1)$, of the polynomial domain $F[\lambda]$ onto $F_{r_0}$ is the prime ideal $(f(\lambda))$. Thus $F_{r_0}$ is a field and hence $N\cap F_{r_0}=0$. Finally, $x\notin A^*$ implies $x_1=x+r_0\notin A^*$, so that $F_{r_0}$ properly contains $F$, contrary to the maximality of $F$. This completes the proof of the theorem.

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