ON THE COEFFICIENTS OF MEROMORPHIC SCHLICHT FUNCTIONS[1]

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1. We denote by S the family of functions

$$f(z) = \frac{1}{z} + a_1 z + a_2 z^2 + \cdots, \qquad |z| < 1, \tag{1}$$

which are univalent in the unit circle, and we set $A_n = \sup |a_n|$, $f(z) \in S$. It has been known a long time that $A_1 = 1$, and it was shown in 1938 by Schiffer [6] and Goluzin [2] that $A_2 = 2/3$. This gave rise to the conjecture $A_n = 2(n+1)^{-1}$ [7] which, because of the simple mapping properties of the "extremal"

$$f_n(z) = z^{-1}(1 + z^{n+1})^{2/(n+1)} = \frac{1}{z} + \frac{2}{n+1} z + \cdots, \tag{2}$$

looked very convincing at the time.

It has, however, recently been shown that this conjecture is false, at least for odd n [1; 3]. Garabedian and Schiffer [1] succeeded, moreover, in proving that the exact value of A_3 is $1/2 + e^{-6}$. While this disposes of the conjecture $(n+1)A_n = 2$ in the case of the general class S, one may nevertheless attempt to save the inequality $(n+1)|a_n| \leqq 2$ by imposing suitable restrictions on the class S. Taking our cue from a somewhat similar situation which arose in the early discussion of the Bieberbach conjecture, we are led to the consideration of two particular sub-classes of S: (a) the class of functions $f(z)$ with real coefficients a_n; (b) the class S_t of functions $f(z)$ which map $|z| < 1$ onto the complement of a point-set starlike with respect to the origin. The case (a), however, is ruled out immediately, since the Garabedian-Schiffer extremal function happens to have real coefficients. We are thus left with the case (b).

It may be remarked that the choice of the origin as the "star center" of the map appears natural in view of the fact that, because of (1),

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,d\theta = \lim_{\rho\to 1} \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{i\theta})\,d\theta = 0,$$

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i.e., the center of gravity of the boundary of the map is at the origin.

In what follows we shall describe a method which, for a given n, may be used to prove that $(n+1)|a_n| \leqq 2$ for $f(z) \in S_t$. The computations which are involved become, however, more and more laborious as n grows larger, and very soon a point is reached beyond which the importance of the result does not seem to be commensurate with the labor required to obtain it. We shall here treat the cases $n=3, 4, 5, 6$, and thus prove

THEOREM I. *Let S_t denote the class of functions* (1) *which are univalent in $|z|<1$ and map $|z|<1$ onto the complement of a point-set starlike with respect to the origin. Then*

$$|a_n| \leqq \frac{2}{n+1} \tag{3}$$

for $n=3, 4, 5, 6$. This inequality is sharp, as shown by the functions (2) (*which belong to S_t*).

We shall supplement Theorem I by the following result which is valid for all n $(n=1, 2, \cdots)$ and which yields information about the functions solving the problem $|a_n| = \text{Max.}$ within the class S_t.

THEOREM II. *A function $f(z)$ which solves the extremal problem*

$$|a_n| = \text{Max.}, \qquad f(z) \in S_t, \qquad n = 1, 2, \cdots \tag{4}$$

is necessarily of the form

$$f(z) = z^{-1} \prod_{\nu=1}^{n+1} (1 - \kappa_\nu z)^{\lambda_\nu} \tag{5}$$

where $|\kappa_\nu| = 1$, $\lambda_\nu \geqq 0$, and $\lambda_1 + \cdots + \lambda_{n+1} = 2$.

Theorem II reduces the problem (4) to an elementary problem in the ordinary calculus. This might be expected to make the proof of (3) easy but—unless the present authors have overlooked something obvious—the proof of (3) *via* Theorem II seems to be difficult even for $n=3$.

2. We now turn to the proof of Theorem I. It is well known that the class S_t is closely related to the class P of functions $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$ which are regular, and have a positive real part in $|z|<1$. If $f(z) \in S_t$, we have

$$\frac{zf'(z)}{f(z)} = -g(z),$$

where $g(z)$ is a suitable function of P. (1) shows that we have, moreover, $b_1=0$. Writing

$$g(z) = 1 + G(z) = 1 + b_2z^2 + b_3z^3 + \cdots \tag{6}$$

and integrating (6), we obtain

$$\begin{aligned} f(z) &= \frac{1}{z} \exp\left\{\int_0^z \frac{G(z)}{z}\, dz\right\} \\ &= \frac{1}{z} \exp\left\{-\frac{1}{2} b_2z^2 - \frac{1}{3} b_3z^3 - \cdots\right\}. \end{aligned} \tag{7}$$

Because of (1), this gives rise to the following identities:

$$-2a_1 = b_2, \qquad -3a_2 = b_3, \tag{8a}$$

$$-4a_3 = b_4 - \frac{1}{2} b_2^2, \qquad -5a_4 = b_5 - \frac{5}{6} b_2b_3, \tag{8b}$$

$$-6a_5 = b_6 - \frac{3}{4} b_2b_4 - \frac{1}{3} b_3^2 + \frac{1}{8} b_2^3, \tag{8c}$$

$$-7a_6 = b_7 - \frac{7}{10} b_2b_5 - \frac{7}{12} b_3b_4 + \frac{7}{24} b_2^2b_3. \tag{8d}$$

The inequalities $|a_1| \leqq 1$, $|a_2| \leqq 2/3$ are trivial consequences of (8a) and the classical inequality $|b_n| \leqq 2$, $n=2, 3, \cdots$. In order to obtain the corresponding inequalities for the higher coefficients a_n, we have to show that the absolute values of the right-hand sides of (8b), (8c), (8d) are bounded by 2. This will be accomplished by means of the following two lemmas.

LEMMA I. *If the functions*

$$1 + \sum_{\nu=1}^{\infty} b_\nu z^\nu, \qquad 1 + \sum_{\nu=1}^{\infty} c_\nu z^\nu$$

belong to P, then the same is true of the function

$$1 + \frac{1}{2} \sum_{\nu=1}^{\infty} b_\nu c_\nu z^\nu.$$

LEMMA II. *Let $h(z)=1+\beta_1z+\beta_2z^2+\cdots$ and $1+G_1(z)=1+b_1'z+b_2'z^2+\cdots$ be functions of P, and set*

$$\gamma_\nu = \frac{1}{2^\nu}\left[1 + \frac{1}{2}\sum_{\mu=1}^{\nu}\binom{\nu}{\mu}\beta_\mu\right], \qquad \gamma_0 = 1. \tag{9}$$

If A_n is defined by

$$\sum_{\nu=1}^{\infty}(-1)^{\nu+1}\gamma_{\nu-1}G_1^{\nu}(z) = \sum_{n=1}^{\infty} A_n z^n, \tag{10}$$

then

$$|A_n| \leqq 2. \tag{11}$$

PROOF OF LEMMA I. We write $g(z)=1+b_1z+b_2z^2+\cdots$, $g_1(z)=1+c_1z+c_2z^2+\cdots$ and we note that the function $g_2(z)=[g_1(z^*)]^*=1+c_1^*z+c_2^*z^2+\cdots$ likewise belongs to P (asterisks denote complex conjugates). If $0<\rho<1$ and $|\kappa|=1$, we thus have

$$\begin{aligned}0 &\leqq \operatorname{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} g(\rho\kappa e^{i\theta}) \operatorname{Re}\,[g_2(\rho e^{i\theta})]d\theta\right\}\\ &= \operatorname{Re}\left\{\frac{1}{4\pi}\int_0^{2\pi} g(\rho\kappa e^{i\theta})[g_2(\rho e^{i\theta}) + g_2^*(\rho e^{i\theta})]d\theta\right\}\\ &= \operatorname{Re}\left\{1+\frac{1}{2}\sum_{\nu=1}^{\infty} b_\nu c_\nu(\rho^2\kappa)^\nu\right\}.\end{aligned}$$

Since $\rho^2\kappa$ may represent any point in the unit circle, this proves Lemma I.

PROOF OF LEMMA II. Since the mapping $w\rightarrow(1+i\alpha w)(w+i\alpha)^{-1}$ (α real) transforms the right half-plane into itself, the function

$$\begin{aligned}h(z) &= \frac{1+i\alpha[1+G_1(z)]}{i\alpha+1+G_1(z)}\\ &= 1-\left(\frac{1-i\alpha}{1+i\alpha}\right)\sum_{\nu=1}^{\infty}(-1)^{\nu+1}\frac{G_1^{\nu}(z)}{(1+i\alpha)^{\nu-1}}\\ &= 1-\left(\frac{1-i\alpha}{1+i\alpha}\right)\sum_{n=1}^{\infty}B_n(\alpha)z^n\end{aligned} \tag{12}$$

will also belong to P. By the classical inequality mentioned further above, we thus have

$$\left|\left(\frac{1-i\alpha}{1+i\alpha}\right)B_n(\alpha)\right| = |B_n(\alpha)| \leqq 2,$$

or, more generally,

$$\left|\sum_{\kappa=1}^{m}\lambda_\kappa B_n(\alpha_\kappa)\right| \leqq 2,$$

where $\lambda_\kappa \geqq 0$, $\lambda_1+\cdots+\lambda_m=1$, and $\alpha_1,\cdots,\alpha_m$ are arbitrary real numbers. By a suitable passage to the limit we obtain

$$\left|\int_0^{2\pi}\sigma(t)B_n[\alpha(t)]dt\right| \leqq 2, \tag{13}$$

where the only restriction on the real function $\alpha(t)$ is the condition that $B_n[\alpha(t)]$ be a continuous function in $0\leqq t\leqq 2\pi$, and $\sigma(t)$ may be any function which is continuous in this interval and such that

$$\sigma(t) \geqq 0, \qquad \int_0^{2\pi}\sigma(t)dt = 1. \tag{14}$$

We now specialize (13) by setting $\alpha(t)=\tan t$. (12) shows that $B_n[\alpha(t)]$ is a linear combination of expressions of the form $[1+i\alpha(t)]^{-\nu}$, $\nu=1, 2, \cdots$, and we thus have to compute the integrals

$$\begin{aligned}\gamma_\nu &= \int_0^{2\pi}\frac{\sigma(t)dt}{(1+i\tan t)^\nu} = \int_0^{2\pi}e^{-i\nu t}\cos^\nu t\sigma(t)dt\\ &= \frac{1}{2^\nu}\int_0^{2\pi}(1+e^{-2it})^\nu\sigma(t)dt\\ &= \frac{1}{2^\nu}\sum_{\mu=0}^{\nu}\binom{\nu}{\mu}\int_0^{2\pi}e^{-2i\mu t}\sigma(t)dt.\end{aligned} \tag{15}$$

If we write $h_1(z)=h(\rho z^2)$ $(0<\rho<1)$ and observe that $h_1(z)\in P$, it is clear that the function $\sigma(t)=(1/2\pi)\mathrm{Re}\ \{h_1(e^{it})\}$ is continuous for $0\leqq t\leqq 2\pi$ and satisfies the conditions (14). This function $\sigma(t)$ may thus be used in (15). Since

$$\int_0^{2\pi}e^{-2i\mu t}\sigma(t)dt = \frac{1}{4\pi}\int_0^{2\pi}e^{-2i\mu t}[h_1(e^{it}) + \overset{*}{h}_1(e^{it})]dt = \frac{1}{2}\beta_\mu\rho^\mu,\ \mu \neq 0,$$

we find, on letting $\rho\to 1$, that γ_ν takes the value (9). A comparison of (12) and (15) shows that

$$\sum_{\nu=1}^{\infty}(-1)^{\nu+1}\gamma_{\nu-1}\overset{\nu}{G}_1(z) = \sum_{n=1}^{\infty}z^n\int_0^{2\pi}\sigma(t)B_n[\alpha(t)]dt.$$

In view of (13), this completes the proof of Lemma II.

3. We now compute the leading coefficients A_n in the expansion (10). Assuming that $b_1'=0$, we obtain

$$A_4 = b_4' - \gamma_1 b_2'^2, \qquad A_5 = b_5' - 2\gamma_1 b_2' b_3', \tag{16a}$$

$$A_6 = b_6' - 2\gamma_1 b_2' b_4' - \gamma_1 b_3'^2 + \gamma_2 b_2'^3, \tag{16b}$$

$$A_7 = b_7' - 2\gamma_1 b_2' b_5' - 2\gamma_1 b_3' b_4' + 3\gamma_2 b_2'^2 b_3'. \tag{16c}$$

By Lemma (2), these coefficients satisfy the inequality (11) if

$1+b_2'z^2+b_3'z^3+\cdots$ is a function of the class P. By virtue of Lemma I, we may also set

$$b_\nu' = \frac{1}{2} b_\nu c_\nu, \tag{17}$$

where $g(z)=1+b_2z^2+b_3z^3+\cdots$ is the function (6) and $H(z)=1+c_1z+c_2z^2+\cdots$ is an arbitrary function in P.

With the values (17) of the b_ν', the expressions (16) become polynomials in the coefficients b_ν, of the same general character as the expressions (8). If, by judicious choice of the functions which give rise to the constants c_ν and γ_ν we can make the coefficients of the corresponding monomials in (8) and (16) identical, Theorem I will be proved. Indeed, A_n is subject to the inequality (11), and the left-hand sides of the identities (8) are of the form $(n+1)a_n$. We proceed to carry out this program for $n=3, 4, 5, 6$.

$n=3$: In view of (8b), (16a) and (17), we have to show that the functions $H(z)=1+c_1z+c_2z^2+\cdots$ and $h(z)=1+\beta_1z+\beta_2z^2+\cdots$ of P may be so chosen that $\gamma_1c_2^2=2$, where γ_1 is given by (9) and $c_4=2$. This can be done in many ways, e.g., $h(z)\equiv 1$ (leading to $\gamma_1=1/2$), $H(z)=(1+z)(1-z)^{-1}=1+2z+2z^2+2z^3+2z^4+\cdots$.

$n=4$: In this case, the conditions to be satisfied are $\gamma_1c_2c_3=5/3$, $c_5=2$, where $\gamma_1=(1+\beta_1/2)/2$. One of the many possible choices is $H(z)=(1+z)(1-z)^{-1}$, $h(z)=5/6+(1-z)(1+z)^{-1}/6=1-z/3+\cdots$.

$n=5$: The conditions are now

$$\gamma_1c_2c_4 = 3/2, \qquad \gamma_1c_3^2 = 4/3, \qquad \gamma_2c_2^3 = 1, \qquad c_6 = 2,$$

where, by (9),

$$\gamma_1 = (1+\beta_1/2)/2, \qquad \gamma_2 = (1+\beta_1+\beta_2/2)/4. \tag{18}$$

It is easily confirmed that this is satisfied by the functions

$$\begin{aligned} H(z) &= \frac{2(2)^{1/2}}{3}\left(\frac{1+z}{1-z}\right)+\left(1-\frac{2(2)^{1/2}}{3}\right)\frac{1+z^2}{1-z^2} \\ &= 1+\frac{4(2)^{1/2}}{3}z+2z^2+\frac{4(2)^{1/2}}{3}z^3+2z^4+\frac{4(2)^{1/2}}{3}z^5+2z^6+\cdots \end{aligned}$$

and

$$\begin{aligned} h(z) &= \frac{1}{2}+\frac{1}{4}\left(\frac{1-z}{1+z}\right)+\frac{1}{4}\left(\frac{1+z^2}{1-z^2}\right) \\ &= 1-\frac{z}{2}-\frac{z^3}{2}+\cdots \qquad \left(\gamma_1=\frac{3}{8},\ \gamma_2=\frac{1}{8}\right), \end{aligned}$$

both of which belong to P. The reader will have noticed that in all these cases the construction of appropriate functions of P is based on the obvious fact that the function $\lambda_1 f_1(z)+\lambda_2 f_2(z)+\cdots+\lambda_m f_m(z)$ will belong to P if $\lambda_\mu \geqq 0$, $\lambda_1+\cdots+\lambda_m=1$, and $f_\mu(z)\in P$ $(\mu=1,2,\cdots,m)$.

$n=6$: (8d), (16c) and (17) show that we must have

$$\frac{7}{5}=\gamma_1 c_2 c_5,\qquad \frac{7}{6}=\gamma_1 c_3 c_4,\qquad \frac{7}{9}=\gamma_2 c_2^2 c_3,\qquad c_7=2.$$

These conditions will be satisfied if

$$\gamma_1=\frac{28}{45},\qquad \gamma_2=\frac{56}{243}\left(\frac{6}{5}\right)^{1/2}, \tag{19}$$

$$c_2=c_5=\frac{3}{2},\qquad c_3=c_4=\frac{3}{2}\left(\frac{5}{6}\right)^{1/2},\qquad c_7=2. \tag{20}$$

That (19) is a possible choice follows from (18) and the fact that the function

$$h(z)=\frac{224}{243}\left(\frac{6}{5}\right)^{1/2}-\frac{44}{45}+\frac{11}{45}\left(\frac{1+z}{1-z}\right)+\left(\frac{78}{45}-\frac{224}{243}\left(\frac{6}{5}\right)^{1/2}\right)\left(\frac{1-z^2}{1+z^2}\right)$$

belongs to P (the sum of the first two numerical terms is positive). A function $H(z)$ which belongs to P and has the coefficients (20) is

$$H(z)=\lambda\left(\frac{1+\omega z}{1-\omega z}+\frac{1+\omega^* z}{1-\omega^* z}\right)+\mu\left(\frac{1+z}{1-z}\right)+(1-2\lambda-\mu)\left(\frac{1+z^7}{1-z^7}\right),$$

where $\omega=\exp\,(2\pi i/7)$ and

$$\lambda=\frac{3\left(1-\left(\frac{5}{6}\right)^{1/2}\right)}{8\left(\cos\frac{\pi}{7}-\cos\frac{3\pi}{7}\right)},$$

$$\mu=\frac{3}{4}+\frac{3\left(1-\left(\frac{5}{6}\right)^{1/2}\right)\cos\frac{3\pi}{7}}{4\left(\cos\frac{\pi}{7}-\cos\frac{3\pi}{7}\right)}$$

(it is easily confirmed that $1-2\lambda-\mu>0$). This disposes of the case $n=6$ and thus completes the proof of Theorem I.

4. The proof of Theorem II is again based on the representation (7). Comparing (1) and (7), we find that

$$-(n+1)a_n = b_{n+1} + F(b_2, b_3, \cdots, b_n),$$

where F is a polynomial of the indicated variables and $b_2, \cdots, b_{n+1}$ are the leading coefficients of the function (6). The problem $|a_n| = \max.$ $(f(z) \in S_t)$ is thus equivalent to the problem $|b_{n+1}+F(b_2, \cdots, b_n)| = \max.$ $(g(z) \in P)$. Since a_n may be assumed to be negative (replacing $f(z)$, if necessary, by $\kappa f(\kappa z)$, $|\kappa|=1$), this problem may also be stated in the form $\operatorname{Re}\{b_{n+1}+F(b_2, \cdots, b_n)\} = \max.$ Because of the compactness of P, there exists a function solving this problem, say $g_0(z) = 1+b_2' z^2+b_3' z^3+ \cdots$. It is evident that the same function will also solve the extremal problem

$$\operatorname{Re}\{b_{n+1}\} = \max., \quad b_\nu = b_\nu', \quad \nu = 2, \cdots, n, \quad b_1 = 0, \quad g(z) \in P.$$

Except for a trivial passage from bounded functions to functions with a positive real part, this is identical with the Pick-Nevanlinna interpolation problem for bounded functions, which is known to be solved by a function $w=g_0(z)$ mapping $|z|<1$ onto the m times covered half-plane $\operatorname{Re}\{w\}>0$, where $m \leqq n+1$ [**4**; **5**]. It is easy to see that $g_0(z)$ must be of the form

$$g_0(z) = \sum_{\nu=1}^{n+1} \sigma_\nu \left(\frac{1+\kappa_\nu z}{1-\kappa_\nu z}\right) \tag{21}$$

where $|\kappa_\nu|=1$, $\sigma_\nu \geqq 0$ $(\nu=1, \cdots, n+1)$, and $\sigma_1+ \cdots +\sigma_{n+1}=1$. Indeed, the reflection principle shows that $g_0(z)$ can have no singularities except for m simple poles on $|z|=1$, say $\kappa_1^*, \cdots, \kappa_m^*$, and that $g_0(z)$ is continued across $|z|=1$ by means of the formula $g_0(z^{*-1})^* = -g_0(z)$. It follows that

$$g_0(z) = c_0 + \sum_{\nu=1}^{n+1} \frac{c_\nu}{1-\kappa_\nu z} = -c_0^* + \sum_{\nu=1}^{n+1} \frac{c_\nu^* \kappa_\nu z}{1-\kappa_\nu z},$$

whence

$$c_0 + \overset{*}{c_0} + \sum_{\nu=1}^{n+1} c_\nu \left\{\frac{1-\dfrac{\overset{*}{c_\nu}}{c_\nu}\kappa_\nu z}{1-\kappa_\nu z}\right\} = 0.$$

This shows that those constants c_ν which are not zero must be real, and that $2\operatorname{Re}\{c_0\}+c_1+c_2+ \cdots +c_{n+1}=0$. Hence,

$$g_0(z) = i \operatorname{Im} \{c_0\} + \sum_{\nu=1}^{n+1} c_\nu \left[\frac{1}{1-\kappa_\nu z} - \frac{1}{2}\right]$$
$$= i \operatorname{Im} \{c_0\} + \frac{1}{2} \sum_{\nu=1}^{n+1} c_\nu \left(\frac{1+\kappa_\nu z}{1-\kappa_\nu z}\right).$$

Because of $g_0(0)=1$, Im $\{c_0\}$ must vanish, and $g_0(z)$ is indeed found to be of the form (21) where the σ_ν are real constants such that $\sigma_1 + \cdots + \sigma_{n+1} = 1$. On $|z|=1$, both $g_0(z)$ and $(1+\kappa_\mu z)(1-\kappa_\mu z)^{-1}$ are pure imaginary. If $\sigma_\mu \neq 0$, both functions tend to $-i\infty$ if $z \to \kappa_\mu^*$ in the positive direction. It follows therefore that

$$\sigma_\mu = \lim_{z \to \kappa_\mu^*} \left(\frac{1-\kappa_\mu z}{1+\kappa_\mu z}\right) g_0(z)$$

must be positive.

The function $f(z)$ which maximizes $|a_n|$ within S_t is related to $g_0(z)$ by means of the identity

$$\frac{zf'(z)}{f(z)} = -g_0(z).$$

Substituting the expression (21), integrating, and writing $\lambda_\nu = 2\sigma_\nu$, we arrive at the formula (5) for the extremal function. This completes the proof of Theorem II.

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