ON MEROMORPHIC FUNCTIONS OF BOUNDED CHARACTERISTIC

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1. **Introduction.** Let $w=f(z)$ be meromorphic in $|z|<1$, and let the characteristic function $T(r,f)$ be bounded, i.e., $T(r,f)=O(1)$ in $|z|<1$. If the radial limit values $f^*(e^{i\theta})=\lim_{r\to 1} f(re^{i\theta})$ (which exist and are finite for almost all $e^{i\theta}$ on $|z|=1$ by the Fatou-Nevanlinna theorem)[2] have modulus one for almost all $e^{i\theta}$, we shall say that $f(z)$ is of *class* (B) in $|z|<1$; if, in addition, a function $f(z)$ of class (B) is bounded, $|f(z)|<1$ in $|z|<1$, we shall say that $f(z)$ is of *class* (A) in $|z|<1$. The principal result concerning functions of class (A) is the following theorem of W. Seidel [**12**, p. 205] which we state for reference in the sequel.

THEOREM 1. *Let $w=f(z)$ be of class* (A) *in $|z|<1$, and take there the value α, $|\alpha|<1$ at most a finite number of times. Then, unless $f(z)$ reduces identically to a finite Blaschke product in $|z|<1$ giving the most general* $(m, 1)$ *conformal map of $|z|<1$ onto $|w|<1$, there exists at least one radius $\theta=\theta_0$, such that $f^*(e^{i\theta_0})=\alpha$.*

It has been shown by G. Hössjer [**5**, p. 55] that the radial limit values of a nonconstant function $f(z)$ of class (A) comprise a set E of measure 2π on $|w|=1$. It follows almost trivially from Theorem 1 that every point of $|w|=1$ belongs to the set E. Indeed, if $e^{i\lambda}$ is an arbitrary point of $|w|=1$, the function

$$g(z) = \exp \frac{f(z)+e^{i\lambda}}{f(z)-e^{i\lambda}}$$

is of class (A) in $|z|<1$ and omits the value 0. By Theorem 1, there exists at least one radius $\theta=\theta_0$, for which $g^*(e^{i\theta_0})=\lim_{r\to 1} g(re^{i\theta_0})=0$, from which it follows that $f^*(e^{i\theta_0})=e^{i\lambda}$.

In §2 we shall prove the analogue of this extension of Hössjer's theorem for functions of class (B). In §3 we shall discuss another class of meromorphic functions which has been studied recently by O.

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[2] For this and the general theory of such meromorphic functions, see [**11**, pp. 208 ff.].

Lehto [**6**; **7**; **8**] and M. Tsuji [**14**]: Let $f(z)$ be meromorphic in $|z|<1$, and let the values which $f(z)$ assumes in $|z|<1$ lie in a domain G whose boundary Γ has positive logarithmic capacity, so that by [**11**, p. 213] $f(z)$ is of bounded characteristic in $|z|<1$. Furthermore, we shall assume that $f^*(e^{i\theta})$ belongs to Γ for almost all $e^{i\theta}$ on $|z|=1$. In what follows, we shall say that a function $f(z)$ with these properties is of *class* (L) in $|z|<1$. Lehto has not only extended the result of Seidel (Theorem 1) by showing that any value a lying in G which is omitted by $f(z)$ is a radial limit value of $f(z)$, but also that any a in G for which the "deficiency"

$$\Phi(a) = \frac{1}{2\pi} \lim_{r\to 1} \int_0^{2\pi} g(f(re^{i\theta}), a)d\theta \tag{1}$$

is positive, where $g(w, a)$ is the Green's function of G with singularity at $w=a$, is also a radial limit value of $f(z)$.

In §3 we shall extend the result of Hössjer to functions of class (L) by showing that every point of Γ which is arcwise accessible from G is also a radial limit value of $f(z)$, and that, except for two special cases, every accessible point of Γ is a radial limit of $f(z)$ infinitely often.

2. We shall assume in this section that $f(z)$ is a nontrivial function of class (B), i.e., that neither $f(z)$ nor $1/f(z)$ reduces to a function of class (A). We shall say that a number α is in the range of $f(z)$ at a point P of $|z|=1$ if α is assumed by $f(z)$ in every neighborhood of P. The number α will be said to be in the range of $f(z)$ in $|z|<1$ if it is assumed infinitely often in $|z|<1$.

THEOREM 2. *If $w=f(z)$ is a nontrivial function of class* (B), *then either every point of $|w|=1$ which is not in the range of $f(z)$ is an asymptotic value of $f(z)$, or else $f(z)$ is a p-valent function mapping $|z|<1$ onto a simply-connected region consisting of the w-plane slit along an arc of $|w|=1$.*

We assume first that the point $e^{i\lambda}$ is not in the range of $f(z)$ in $|z|<1$, i.e., $e^{i\lambda}$ is assumed at most finitely often by $f(z)$ in $|z|<1$. Then the function

$$\phi(z) = \exp \frac{f(z) + e^{i\lambda}}{f(z) - e^{i\lambda}} \tag{2}$$

is analytic in some annulus $\rho<|z|<1$, and omits the value 0. Since, by the Riesz-Nevanlinna Theorem [**11**, p. 209], $\liminf_{r\to 1} |f(z)-e^{i\lambda}|$ can be zero on at most a set of measure zero of $|z|=1$, the function

$\phi(z)$ has the property that its radial limits exist and have modulus 1 almost everywhere on $|z|=1$.

Now if $e^{i\lambda}$ is not assumed at all in $|z|<1$ by $f(z)$, the function $\phi(z)$ in (2) is analytic in $|z|<1$ and omits there the value 0. Since $\lim_{r\to 1}|\phi(re^{i\theta})|=1$ almost everywhere on $|z|=1$, it follows from a theorem of Lohwater [**9**, p. 248] that either 0 or ∞ is an asymptotic value of $\phi(z)$, so that $e^{i\lambda}$ is an asymptotic value of $f(z)$.

Suppose next that $e^{i\lambda}$ is actually assumed by $f(z)$ in $|z|<1$, but by hypothesis, only finitely often. From a recent extension of Schwarz's reflection principle by Lohwater [**10**] we have that a necessary and sufficient condition that $\phi(z)$ can be continued analytically beyond $|z|=1$ is that $\phi(z)$ admits neither 0 nor ∞ as an asymptotic value. If $f(z)$ is not analytic at every point of $|z|=1$, there exists an arc L, lying in $|z|<1$ and terminating at a point $e^{i\theta_0}$ such that, as $z\to e^{i\theta_0}$ along L, $\phi(z)$ tends to 0 or ∞, whence $f(z)\to e^{i\lambda}$ along L, so that $e^{i\lambda}$ is an asymptotic value of $f(z)$.

On the other hand, if $\phi(z)$ is analytic on $|z|=1$, then $f(z)$ can have only a finite number of zeros and poles in $|z|<1$. If for example, $f(z)$ has an infinite number of zeros in $|z|<1$, let $\{z_k\}$, $k=1, 2, \cdots$, denote a subsequence of these zeros converging to some point $e^{i\theta_0}$ on $|z|=1$. On this subsequence we must have $\lim_{k\to\infty}\phi(z_k)=\phi(e^{i\theta_0})=e^{-1}$ which is a contradiction since $\phi(z)$ must be analytic with modulus 1 everywhere on $|z|=1$. We consider, finally, the Poisson-Stieltjes representation [**11**, p. 201] of $w=f(z)$.

$$(3)\quad f(z)=\prod_{j=1}^{m}\frac{\alpha_j-z}{1-\bar{\alpha}_j z}\prod_{k=1}^{n}\frac{1-\bar{\beta}_k z}{\beta_k-z}\exp\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)+i\gamma\right],$$

where α_j and β_k are the zeros and poles, respectively of $f(z)$, γ is a real constant, and $\mu(t)$ is of bounded variation on $0\leqq t\leqq 2\pi$.

Since the finite products in (3) have modulus 1 everywhere on $|z|=1$, it follows from an argument identical to that used in [**9**, p. 246] that $\mu(t)$ is identically constant, so that $f(z)$ reduces to a quotient of two finite Blaschke products,

$$(4)\qquad f(z)=e^{i\gamma}\prod_{j=1}^{m}\frac{\alpha_j-z}{1-\bar{\alpha}_j z}\prod_{k=1}^{n}\frac{1-\bar{\beta}_k z}{\beta_k-z},\qquad n, m\geqq 1.$$

It cannot happen that the domain G in the w-plane onto which $f(z)$ maps $|z|<1$, is multiply connected, for otherwise $f(z)$ must then assume infinitely often in $|z|<1$ every value of G; in particular, $e^{i\lambda}$ would then be in the range of $f(z)$, a contradiction. It is easy to see that the simply-connected region G can be mapped onto the circle

$|t|<1$; if $t=F(w)$, $F(0)=0$ is a univalent function which effects such a map, then the function $t=F(f(z))$ is analytic of class (A) in $|z|<1$ with m zeros. Since zero is not an asymptotic value of $F(f(z))$, it follows from Theorem 1 that $F(f(z))$ gives an $(m, 1)$ conformal mapping of $|z|<1$ onto $|t|<1$, so that $f(z)$ in (4) gives an $(m, 1)$ conformal mapping of $|z|<1$ onto the region G which consists of the w-plane slit along an arc of $|w|=1$.

We remark that this theorem is similar to a theorem of Lehto [**7**, p. 12]; the example given below will show, however, that the class (B) is not contained in the class (L).

3. **Functions of class** (L). We assume that values which $w=f(z)$ assumes in $|z|<1$ lie in some domain G of the w-plane whose boundary Γ has positive capacity and $f^*(e^{i\theta})$ belongs to Γ for almost all $e^{i\theta}$ on $|z|=1$.

THEOREM 3. *Let $w=f(z)$ be of class* (L) *in $|z|<1$ with respect to a domain G in the w-plane whose boundary Γ has positive capacity. Let $f^*(e^{i\theta})$ belong to Γ for almost all $e^{i\theta}$ on $|z|=1$. Then every arcwise accessible point of Γ is a radial limit value of $f(z)$.*

Let D be that subdomain of G which consists of the values which $f(z)$ assumes in $|z|<1$. It has been shown by Lehto [**8**, p. 97] that the set S of points of G not assumed by $f(z)$ is of capacity zero; obviously $G=D\cup S$.

From the fact that a set S of capacity zero cannot separate the plane, it is a simple consequence that if a point $\alpha\in\Gamma=\mathrm{Fr}\ (G)$ is arcwise accessible from G it is arcwise accessible from D. Indeed, let L be a Jordan arc lying in G and terminating at α. Let κ_n, $n=1, 2, \cdots$, be a set of circular neighborhoods lying in G with centers C_n on L and radii r_n such that as $n\to\infty$, $C_n\to\alpha$ and $r_n\to 0$, and such that $\kappa_n\cap\kappa_{n+1}$ is not empty for any n. Since $\kappa=\bigcup_{n=1}^{\infty}\kappa_n$ is open and covers L, there exists also another Jordan arc J disjoint from L, lying in κ and terminating at α. Next pick two sequences of points, l_n belonging to $L\cap D$ and j_n belonging to $J\cap D$, such that as $n\to\infty$, $l_n\to\alpha$ and $j_n\to\alpha$. This is clearly possible since S is of capacity zero and thus can contain no arc. Because S cannot separate the plane, we are able to join l_n to j_n by an arc γ_n lying in D and to pick a point w_n on γ_n, $l_n\neq w_n\neq j_n$. We join w_n to w_{n+1} by an arc K_n: $w=w(t)$, $1/(n+1)<t\leq 1/n$ such that K_n lies in D. Since $K=\bigcup_{n=1}^{\infty}K_n$ is an arc: $w=w(t)$, $0<t\leq 1$, lying in D and terminating at α, we have that α is arcwise accessible from D. Let L_α be an arc in D which terminates at the point α. Let $w=w(\zeta)$, $w(0)=f(0)$ be a function which maps the circle

$|\zeta|<1$ conformally onto the universal covering surface $\mathfrak{R}_D$ of D; since Γ has positive capacity, such a mapping will exist and will be of bounded characteristic in $|\zeta|<1$. If we denote by $\zeta=\zeta(w)$ that branch of the inverse function of $w=w(\zeta)$ which vanishes at $w=f(0)$, the function $\zeta=F(z)=\zeta(f(z))$ can be continued along any path lying in $|z|<1$, so that, by the monodromy theorem, $F(z)$ will be a single-valued analytic function of class (A) in $|z|<1$.

There exists at least one point $e^{i\beta}$ on $|\zeta|=1$ and an arc L_β of $|\zeta|<1$ terminating at $e^{i\beta}$ such that the image of L_β under $w=w(\zeta)$ is L_α. Since $w=w(\zeta)$ omits at least three values, it follows from an extension of a theorem of Lindelöf [2, p. 96] that the $\lim_{\zeta\to e^{i\beta}} w(\zeta)=\alpha$ exists uniformly in the angle V_δ: $|\arg(1-\zeta e^{-i\beta})|\leqq(\pi/2)-\delta$, for any $\delta>0$. Now $\zeta=F(z)$ as a function of class (A) assumes in $|z|<1$ all values in $|\zeta|<1$ except for a set S' of capacity zero [3, p. 111]. Furthermore $F'(z)$ is an analytic function in $|z|<1$ and, as such, possesses at most a countable number of zeros $\{p_n\}$. If we denote $\bigcup_n F(p_n)=Q$ we see that it is possible to find a second arc L'_β terminating at $e^{i\beta}$ and lying inside V_δ such that L'_β does not pass through any points of the set $S'\cup Q$. It now follows from a well-known method (see, for example, [1, p. 230]) that a branch of the inverse function of $F(z)$ may be continued along L'_β, so that there exists an arc L_z lying in $|z|<1$ and terminating at a point $e^{i\theta_0}$ of $|z|=1$ such that, as $z\to e^{i\theta_0}$ along L_z, $\zeta\to e^{i\beta}$ along L'_β. From the identity $w(\zeta)=w(F(z))=f(z)$, it follows that $f(z)\to\alpha$ as $z\to e^{i\theta_0}$ along L_z, so that by the extension of Lindelöf's theorem mentioned above $\lim_{r\to 1} f(re^{i\theta_0})=\alpha$.

We remark that every arcwise accessible point α of Γ is a radial limit of $f(z)$ infinitely often except for two cases: (1) whenever G is simply-connected and $f(z)$ effects an $(m, 1)$ conformal mapping of $|z|<1$ onto G; in this case, each arcwise accessible point $\alpha\in\Gamma$ is a radial limit exactly m times; (2) whenever G is doubly-connected and one boundary component reduces to a point, and $f(z)$ maps $|z|<1$ conformally onto the covering surface of G; in this case the degenerate boundary component will be a radial limit finitely often, while all other arcwise accessible boundary points of G will be radial limits of $f(z)$ infinitely often. As an example of case (2) we exhibit the function $w=\exp((z^n+1)/(z^n-1))$ which maps $|z|<1$ in an $(n, 1)$ conformal way onto the covering surface of the disc $|w|<1$ punctured at $w=0$. Here the boundary Γ of G consists of the circumference $|w|=1$ and the point $w=0$; the degenerate boundary component, $w=0$, is a radial limit only at the nth roots of unity, while all points of $|w|=1$ are radial limits infinitely often.

We conclude this section by exhibiting a function $f(z)$ which is of class (B) but not of class (L); $f(z)$ is the quotient of two Blaschke products $B_1(z)/B_2(z)$, where $B_1(z)$ has an infinite number of zeros $\{z_n\}$ on the oricycle $r=\cos\theta$, $(0<\theta\leqq\pi/2)$, and where $B_2(z)$ has its zeros at the conjugate points $\{\bar{z}_n\}$. We shall show that $f(z)\to 0$ as $z\to 1$ along the curve $r=\cos\theta$ $(0<\theta\leqq\pi/2)$ and that $f(z)\to\infty$ as $z\to 1$ along $r=\cos\theta$ $(-\pi/2\leqq\theta<0)$. It will then follow from a theorem of Lindelöf [11, p. 67] that $f(z)$ must assume all complex values—with two possible exceptions—infinitely often in the region $|z-1/2|<1/2$, so that the values assumed by $f(z)$ in $|z|<1$ cannot lie in a domain whose boundary has positive capacity.

We shall simplify the construction of the Blaschke products by considering their analogues in the half-plane $\Re(z)>0$; by conformal mapping, the properties described in the last paragraph will then hold in the unit circle. We consider the function

$$b_1(z) = \prod_{n=1}^{\infty} \frac{z - z_n}{z + \bar{z}_n}$$

where $\Re(z_n)>0$; if $\sum_{n=1}^{\infty} \Re(1/z_n)<\infty$ then $b_1(z)$ is the analogue of the Blaschke product in the right half-plane (cf. [13, p. 142]). We choose the zeros of $b_1(z)$ to be the numbers $z_n=1+in^t$, where $1/2<t<1$; clearly $\sum_{n=1}^{\infty} \Re(1/z_n)$ converges with this choice of t. The zeros of $b_1(z)$ lie on the line $\Re(z)=1$, $\Im(z)>0$. We form the function

$$b_2(z) = \prod_{n=1}^{\infty} \frac{z - \bar{z}_n}{z + z_n},$$

whose zeros are the complex conjugates of the zeros of $b_1(z)$, and consider the function

$$f(z) = \frac{b_1(z)}{b_2(z)} = \prod_{n=1}^{\infty} \frac{z - z_n}{z + \bar{z}_n}\,\frac{z + z_n}{z - \bar{z}_n}. \tag{5}$$

We show that $\lim_{y\to+\infty} f(1+iy)=0$ and $\lim_{y\to-\infty} f(1+iy)=\infty$. If $z=1+iy$, $z_n=1+in^t$, each factor

$$\frac{z - z_n}{z + \bar{z}_n}\,\frac{z + z_n}{z - \bar{z}_n} = \frac{i(y - n^t)}{2 + i(y - n^t)}\cdot\frac{2 + i(y + n^t)}{i(y + n^t)} \tag{6}$$

is in modulus less than 1 for all n and for $y>0$, and the modulus of $w=f(1+iy)$ is less than any one of its factors. Now if $n^t\leqq y\leqq(n+1)^t$, we have that

$$\frac{y - n^t}{|2 + i(y - n^t)|} \leqq \frac{y - n^t}{2} \leqq \frac{(n + 1)^t - n^t}{2},$$

and $(n+1)^i - n^i \to 0$ as $n \to \infty$. Since $\lim_{n\to\infty} (i(y+n^i)/2+i(y+n^i)) = 1$, it follows that the expression in (6) tends to 0 as $y \to +\infty$. Similarly, it can be shown that if $y<0$ each factor (6) of (5) has modulus greater than 1 and tends to ∞ as $y \to -\infty$. Hence $f(1+iy)$ has the properties described above, and the example is completed.

Added December 6, 1955: After this paper was submitted, Theorem 3 has appeared in a later paper of Lehto, Annales Academiae Scientiarum Fennicae no. 177 (1954) p. 45; the author has heard that a third proof of this theorem has also been made by M. Ohtsuka. There has also appeared a paper of K. Noshiro, Proc. Nat. Acad. Sci. U.S.A. vol. 41 (1955) pp. 398–401 containing generalizations of these results.

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