

REFERENCES

1. R. P. Boas, *A theorem on analytic functions of a real variable*, Bull. Amer. Math. Soc. vol. 41 (1935) pp. 233–236.
2. Alberto Calderon and Allen Devinatz, *Sur certaines courbes dans l'espace de Hilbert*, C. R. Acad. Sci. Paris vol. 241 (1955) pp. 539–541.
3. ———, *Sur certaines courbes à courbure constante dans l'espace de Hilbert*, C. R. Acad. Sci. Paris vol. 241 (1955) pp. 586–587.
4. Carl-Gustave Esseen, *Fourier analysis of distribution functions*, Acta Math. vol. 77 (1945) pp. 1–125.

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ON A SERIES OF RAINVILLE INVOLVING LEGENDRE POLYNOMIALS

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1. The object of this paper is to obtain some relations involving Legendre polynomials with the help of a series given by E. D. Rainville. The results are believed to be new.

2. We start with the series given by E. D. Rainville

$$(2.1) \quad P_n(\cos \alpha) = \left(\frac{\sin \alpha}{\sin \beta} \right)^n \sum_{k=0}^n c_{n,k} \left[\frac{\sin (\beta - \alpha)}{\sin \alpha} \right]^{n-k} P_k(\cos \beta).$$

Putting $\beta = 2\alpha$ and $\cos 2\alpha = x$, we get

$$(2.2) \quad 2^{n/2}(1+x)^{n/2}P_n\left(\left(\frac{1+x}{2}\right)^{1/2}\right) = \sum_{k=0}^n c_{n,k}P_k(x).$$

From (2.2) and the orthogonal property

$$(2.3) \quad \begin{aligned} \int_{-1}^1 (1+x)^{n/2}P_r(x)P_n\left(\left(\frac{1+x}{2}\right)^{1/2}\right)dx \\ = \frac{c_{n,\gamma}}{2^{n/2-1}(2\gamma+1)}, & \quad 0 \leq \gamma \leq n, \\ = 0, & \quad r > n. \end{aligned}$$

Using (2.3) with Adams' expansion (*Modern analysis*, p. 331) for

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$P_p(x)P_q(x)$, where p and q are positive integers and $q \leq p$,

$$\begin{aligned}
 & \int_{-1}^1 (1+x)^{n/2} P_n\left(\left(\frac{1+x}{2}\right)^{1/2}\right) P_p(x) P_q(x) dx \\
 (2.4) \quad &= \frac{1}{2^{n/2-1}} \sum_{\gamma=0}^q \frac{A_{p-\gamma} A_\gamma A_{q-\gamma} C_{n,p+q-2\gamma}}{A_{p+q-\gamma} (2p+2q-2\gamma+1)} = \frac{(1/2)_p (1/2)_q n!}{(1/2)_{p+q+1} p! q! 2^{n/2}} \\
 & \times {}_6F_5 \left[\begin{matrix} -q, -p, \frac{1}{2}-p-q, -\frac{1}{2}(p+q), -\frac{1}{2}(p+q-1), \frac{1}{2}; \\ -q-p, \frac{1}{2}-p, \frac{1}{2}-q, \frac{1}{2}(n+1-p-q), \frac{1}{2}(n+2-p-q); \end{matrix} 1 \right]
 \end{aligned}$$

where

$$A_\gamma = \frac{1 \cdot 3 \cdot 5 \cdots (2\gamma - 1)}{\gamma!},$$

$$(a)_\gamma = a(a+1) \cdots (a+\gamma-1), \quad (a)_0 = 1,$$

and $n \geq (p+q)$.

Also when $s = p_1 + p_2 + \cdots + p_\gamma$, following the method of Dr. N. G. Shabde (1945) we obtain,

$$\begin{aligned}
 (2.5) \quad & \int_0^1 x^{s+1} P_{p_1}(x) P_{p_2}(x) \cdots P_{p_\gamma}(x) P_s(2x^2 - 1) dx \\
 &= \frac{(s!)^2}{2^{s+1}(2s+1)!} \prod_{t=1}^\gamma \frac{(2p_t)!}{p_t! p_t!}.
 \end{aligned}$$

With Gormley's result,

$$x^m = \sum_{s=0}^{\lfloor m/2 \rfloor} \frac{(2m-4s+1)m!\Gamma(1/2)}{2^{m+1}s!\Gamma(m-s+3/2)} P_{m-2s}(x)$$

we find that,

$$\begin{aligned}
 (2.6) \quad & \int_{-1}^1 x^m (1+x)^{s/2} P_{p_1}\left(\left(\frac{1+x}{2}\right)^{1/2}\right) \cdots P_{p_\gamma}\left(\left(\frac{1+x}{2}\right)^{1/2}\right) \\
 & \quad \times P_{p_{\gamma+1}}(x) \cdots P_{p_{\gamma+n}}(x) P_l(x) dx \\
 &= \frac{m!(l!)^2 \Gamma(1/2)(2m+1)}{2^{m+s/2} \Gamma(m+3/2)(2l+1)!} \prod_{t=0}^{\gamma+n} \frac{(2p_t)!}{p_t! p_t!}
 \end{aligned}$$

where

$$p_1 + p_2 + \cdots + p_\gamma = s, \quad p_0 = m$$

and

$$s + m + p_{\gamma+1} + \cdots + p_{\gamma+n} = l.$$

Differentiating both sides of (2.2) r -times and using Grosswald's formula (1950),

$$(2.7) \quad \begin{aligned} & \left[\frac{d^r}{dx^r} \left\{ (1+x)^{n/2} P_n \left(\left(\frac{1+x}{2} \right)^{1/2} \right) \right\} \right]_{x=1} \\ &= \sum_{k=\gamma}^n c_{n,k} \frac{(k+\gamma)!}{2^{n/2+\gamma} \gamma! (k-\gamma)!} \\ &= 2^{\gamma-n/2} \left(\frac{1}{2} \right)_\gamma c_{n,\gamma} {}_2F_1 \left[\begin{matrix} -n+\gamma, & 1+2\gamma; \\ & 1+\gamma; \end{matrix} \begin{matrix} -1 \\ \end{matrix} \right]. \end{aligned}$$

Using Neumann's formula with (2.2) we get,

$$(2.8) \quad \int_{-1}^1 \frac{(1+x)^{n/2} P_n \left(\left(\frac{1+x}{2} \right)^{1/2} \right)}{y-x} dx = 2^{1-n/2} \sum_{k=0}^n c_{n,k} Q_k(y)$$

where $y > 1$.

Also (*Higher transcendental functions*, p. 171, result (23))

$$(2.9) \quad \begin{aligned} & \int_0^1 x^\sigma (1+x)^{n/2} P_n \left(\left(\frac{1+x}{2} \right)^{1/2} \right) F_q \left(\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} \begin{matrix} \pm x^2 t^2 \\ \end{matrix} \right) dx \\ &= \sum_{k=0}^n \frac{c_{n,k} \Gamma(1+\sigma/2) \Gamma(1/2+\sigma/2)}{2^{1+n/2} \Gamma(1+\sigma/2-k/2) \Gamma(3/2+\sigma/2+k/2)} \\ & \quad \times {}_{p+2}F_{q+2} \left(\begin{matrix} \alpha_1, \dots, \alpha_p, 1+\sigma/2, 1/2+\sigma/2; \\ \beta_1, \dots, \beta_q, 1+\sigma/2-k/2, 3/2+\sigma/2+k/2; \end{matrix} \begin{matrix} \pm t^2 \\ \end{matrix} \right) \end{aligned}$$

$|t| < 1$ when $p = q + 1$, and the hypergeometric series should be terminating when $p > q + 1$.

We can get many particular cases of the result (2.9).

We have due to Fred Braffman (1951, p. 944) the following generating function for Legendre Polynomials $P_n(x)$

$$(2.10) \quad \begin{aligned} & {}_2F_1 \left[\begin{matrix} a, 1-a; \\ 1; \end{matrix} \begin{matrix} (1-t-\rho) \\ 2 \end{matrix} \right] \times {}_2F_1 \left[\begin{matrix} a, 1-a; \\ 1; \end{matrix} \begin{matrix} (1+t-\rho) \\ 2 \end{matrix} \right] \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (1-a)_n}{n! n!} P_n(x) t^n \end{aligned}$$

with $\rho = (1 - 2xt + t^2)^{1/2}$.

The series in (2.10) is absolutely and uniformly convergent for x , when $|t| < 1$, and therefore,

$$\begin{aligned}
 (2.11) \quad & \int_{-1}^1 (1+x)^{n/2} {}_2F_1 \left[\begin{matrix} a, 1-a; \\ 1; \end{matrix} \frac{(1-t-\rho)}{2} \right] \\
 & \times {}_2F_1 \left[\begin{matrix} a, 1-a; \\ 1; \end{matrix} \frac{(1+t-\rho)}{2} \right] P_n \left(\left(\frac{1+x}{2} \right)^{1/2} \right) dx \\
 & = \sum_{k=0}^n \frac{c_{n,k}(a)_k(1-a)_k}{2^{n/2-1} k! k!(2k+1)} t^k \\
 & = 2^{1-n/2} {}_4F_3 \left[\begin{matrix} -n, a, 1-a, 1/2; \\ 1, 1, 3/2; \end{matrix} -t \right]
 \end{aligned}$$

with

$$\rho = 1 - 2xt + t^2)^{1/2}.$$

When in (2.11) $a = -m$, where m is a positive integer we obtain,

$$\begin{aligned}
 (2.12) \quad & \int_{-1}^1 (1+x)^{n/2} P_m(t+\rho) P_m(t-\rho) P_n \left(\left(\frac{1+x}{2} \right)^{1/2} \right) dx \\
 & = \sum_{k=0}^n \frac{c_{n,k}(-m)_k(m+1)_k}{2^{n/2-1} k! k!(2k+1)} t^k \\
 & = 2^{1-n/2} {}_4F_3 \left[\begin{matrix} -n, -m, 1+m, 1/2; \\ 1, 1, 3/2; \end{matrix} -t \right]
 \end{aligned}$$

with

$$\rho = (1 - 2xt + t^2)^{1/2}.$$

Again the series (Fred Brafman, 1951, p. 946),

$$(2.13) \quad (1-xt)^{-b} {}_2F_1 \left[\begin{matrix} b/2, (b+1)/2; \\ 1; \end{matrix} \frac{t^2(x^2-1)}{(1-xt)^2} \right] = \sum_{n=0}^{\infty} \frac{(b)_n P_n(x)}{n!} t^n$$

is uniformly and absolutely convergent, when $|t| < 1$, therefore,

$$\begin{aligned}
 & \int_{-1}^1 (1+x)^{n/2} (1-xt)^{-b} \\
 & \quad \times {}_2F_1 \left[\begin{matrix} b/2, (b+1)/2; \\ 1; \end{matrix} \frac{t^2(x^2-1)}{(1-xt)^2} \right] P_n \left(\left(\frac{1+x}{2} \right)^{1/2} \right) dx \\
 (2.14) \quad & = \sum_{k=0}^n \frac{c_{n,k}(b)_k}{2^{n/2-1} k! (2k+1)} t^k = 2^{1-n/2} {}_3F_2 \left[\begin{matrix} -n, b, 1/2; \\ 1, 3/2; \end{matrix} -t \right].
 \end{aligned}$$

3. In (2.2) putting $((1+x)/2)^{1/2}=y$, we get

$$(3.1) \quad 2^n y^n P_n(y) = \sum_{k=0}^n (-1)^k c_{n,k} P_k(1-2y^2).$$

We have also (Mitra)

$$(3.2) \quad \int_0^1 P_n(1-2y^2) J_0(yx) y dy = \frac{1}{x} J_{2n+1}(x).$$

Using (3.2) with (3.1) we get,

$$(3.3) \quad \int_0^1 y^{n+1} J_0(yx) P_n(y) dy = \frac{1}{2^n x} \sum_{k=0}^n (-1)^k c_{n,k} J_{2k+1}(x).$$

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REFERENCES

1. Fred Brafman, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 942-949.
2. A. Erdélyi, *Higher transcendental functions*, 1953.
3. P. G. Gormley, J. London Math. Soc. vol. 9 pp. 149-152.
4. E. Grosswald, Proc. Amer. Math. Soc. vol. 1 (1950) p. 553.
5. Mitra, Proceedings of the Edinburgh Mathematical Society (2) vol. 4 Part III, p. 111.
6. Earl D. Rainville, Bull. Amer. Math. Soc. vol. 51 (1945) p. 268.
7. N. G. Shabde, Proceedings of the Benares Mathematical Society vol. 7 (1945) pp. 1-2.
8. E. T. Whittaker and G. N. Watson, *Modern analysis*, 4th ed., 1940.