10. J. Todd, *The conditions of the finite segments of the Hilbert matrix, contributions to the solutions of systems of linear equations and the determination of eigenvalues,* National Bureau of Standards Applied Mathematics Series vol. 39 (1950) pp. 109–116.

11. H. Wielandt, *Unzerlegbare, nicht negative Matrizen,* Math. Zeit. vol. 52 (1950) pp. 642–648.

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THE GIBBS PHENOMENON FOR BOREL MEANS

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1. Statement of result.

We prove here the following

THEOREM. *Let $B_x(t)$ denote the xth Borel exponential or integral mean of the Fourier series*

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n}. \tag{1}$$

Then, for given T, $0 \leqq T \leqq \infty$,

$$\lim_{x\to\infty} B_x(t_x) = \int_0^T \frac{\sin v}{v}\, dv, \tag{2}$$

where[1]

$$t_x \to 0+ \quad and \quad xt_x \to T. \tag{3}$$

Thus, the Borel means display the same Gibbs phenomenon and have the same Gibbs ratio as classic convergence, even achieving this ratio for the same value, π, of the parameter T. Except for the last assertion, the same is true (as O. Szász has shown [5; 6]) of the generalized Euler means E_r, $0<r<1$, all of which are equivalent to the Borel summation method for Fourier series and whose Lebesgue

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[1] The assumption that $t_x \to 0+$ is redundant except when T is infinite. The more restrictive condition that $nt_n^2 \to 0$, which, again, is needed only when $T=\infty$, is imposed by O. Szász in his first discussion of the corresponding problem for generalized Euler means [5]. The analogous restriction here would also simplify the technical details of the proof, as shown in §3.

constants display behavior similar to that of the corresponding constants in the Borel case [**2**; **3**; **4**].

However, there is one variation in the pattern. We remark first that for the E_r means the upper limit of the integral in (2) is replaced by rT. Hence the "Gibbs limit" (2) for the Borel means does not equal the corresponding limit for any of the E_r means (except for the trivial case $r=1$ in which the means reduce to the ordinary partial sums for convergence), whereas in the case of the Lebesgue constants, the logarithmic term is the same in all cases and for the Borel and $E_{1/2}$ (the original Euler transformation) methods even the constant terms are identical.

2. **Proof of theorem.** We consider the exponential means first. As usual, the xth Borel exponential mean of a sequence $\{s_n(t)\}$ is defined to be $B_x(t)$, where

$$B_x(t) = e^{-x}\sum_{n=0}^{\infty} s_n(t)\frac{x^n}{n!}\cdot \tag{4}$$

Let $s_0(t)=0$ and take $s_n(t)$ to be the nth partial sum of the series (1), $0<t<\pi$. We use the following standard integral representation of $s_n(t)$,

$$s_n(t) = -\frac{t}{2} + \int_0^t \frac{\sin(n+1/2)u}{2\sin(u/2)}\,du, \qquad n = 0, 1, 2, \cdots \tag{5}$$

and the series evaluation

$$\sum_{n=0}^{\infty}\left[\sin\left(n+\frac{1}{2}\right)u\right]\frac{x^n}{n!} = [\exp(x\cos u)]\left[\sin\left(x\sin u+\frac{1}{2}u\right)\right]. \tag{6}$$

Now we substitute (5) in (4), change the order of summation and integration, and then use (6). Next we replace $1-\cos u$ by $2\sin^2(u/2)$ and effect a change of variable from $u/2$ to u. This gives

$$B_x(t_x) + \frac{1}{2}t_x = \int_0^{t_x/2}\exp(-2x\sin^2 u)\frac{\sin(x\sin 2u+u)}{\sin u}\,du.$$

This integral can be simplified exactly as in §3 of [**3**], and so the detailed justifications are omitted. First we replace the denominator $\sin u$ by u, then $\exp(-2x\sin^2 u)$ by $\exp(-2xu^2)$ and finally $\sin(x\sin 2u+u)$ by $\sin(2x+1)u$. All errors are $o(1)$ as x becomes infinite. In sum, we have

$$B_x(t_x) = \int_0^{t_x/2}\exp(-2xu^2)\frac{\sin(2x+1)u}{u}\,du + o(1), \qquad x\to\infty, \tag{7}$$

since $t_x\to 0$.

The next stage is the elimination of the exponential. For this purpose we define

$$
(8)\qquad \begin{aligned} D_x(t_x) &\equiv \int_0^{t_x/2} \frac{1-\exp(-2xu^2)}{u}\sin(2x+1)u\,du \\ &\equiv \int_0^{t_x/2} f_x(u)\sin(2x+1)u\,du, \end{aligned}
$$

and follow the procedure found in §§5 and 6 of [3].

Use is made of the obvious inequality

$$
(9)\qquad 1-e^{-y} \leqq y, \qquad y \geqq 0.
$$

Integrating $D_x(t_x)$ by parts gives

$$
\begin{aligned} |D_x(t_x)| &\leqq \left| f_x\left(\frac{1}{2}t_x\right)\int_0^{t_x/2}\sin(2x+1)u\,du\right| \\ &\quad + \int_0^{t_x/2}|f_x'(u)|\left|\int_0^u \sin(2x+1)y\,dy\right|du \\ &= f_x\left(\frac{1}{2}t_x\right)O(1/x) + O(1/x)\int_0^{t_x/2}|f_x'(u)|\,du, \end{aligned}
$$

and so, from (9) and (3),

$$
(10)\qquad \begin{aligned} |D_x(t_x)| &= O(t_x) + O(1/x)\int_0^{t_x/2}|f_x'(u)|\,du \\ &= O(1/x)\int_0^{t_x/2}|f_x'(u)|\,du + o(1). \end{aligned}
$$

By (6.7) of [3] and (3) above, the integral on the right in (10) is $O(x^{1/2})$, and so,

$$
(11)\qquad D_x(t_x) = O(x^{-1/2}) + o(1) = o(1).
$$

Thus,

$$
\begin{aligned} B_x(t_x) &= \int_0^{t_x/2}\frac{\sin(2x+1)u}{u}\,du + o(1) = \int_0^{xt_x+t_x/2}\frac{\sin v}{v}\,dv + o(1) \\ &= \int_0^{T}\frac{\sin v}{v}\,dv + o(1), \end{aligned}
$$

completing the proof of the theorem for the case of the exponential means.

For the xth integral mean, $B_x^*(t)$, the result follows on combining the result for the exponential case with a familiar identity [1, p. 182] connecting the two Borel means of a series $\sum a_n$:

$$B_x(t) = e^{-x} \sum a_n \frac{x^n}{n!} + B_x^*(t). \tag{12}$$

Here $a_0 = 0$ and $a_n = n^{-1} \sin nt$; so

$$\left| e^{-x} \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \right| \leqq e^{-x} \sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n}{n!} \to 0, \qquad x \to \infty,$$

uniformly in t, whence the proof is complete.

REMARK. The identity (12) implies similarly that the xth Lebesgue constant for the Borel integral mean differs from that for the exponential mean by $o(1)$.

3. **A special case.** The proof of our result can be simplified if we replace the hypothesis that $t_x \to 0+$ by the somewhat more restrictive requirement that $xt_x^2 \to 0$, a condition which is satisfied whenever T is finite (since $xt_x \to T$).

Under this new hypothesis, the calculations following (8) and leading to (11) can be replaced by the following:

$$\begin{aligned} | D_x(t_x) | &\leqq \int_0^{t_x/2} \frac{1 - \exp(-2xu^2)}{u} du \\ &\leqq \int_0^{t_x/2} \frac{2xu^2}{u} du = \frac{1}{4} xt_x^2 = o(1), \end{aligned}$$

where (9) is used to justify the second inequality.

REFERENCES

1. G. H. Hardy, *Divergent series*, Oxford, 1949.

2. Arthur E. Livingston, *The Lebesgue constants for Euler (E, p) summation of Fourier series*, Duke Math. J. vol. 21 (1954) pp. 309–314.

3. Lee Lorch, *The Lebesgue constants for Borel summability*, Duke Math. J. vol. 11 (1944) pp. 459–467.

4. ———, *The Lebesgue constants for (E, 1) summation of Fourier series*, Duke Math. J. vol. 19 (1952) pp. 45–50.

5. Otto Szász, *On the Gibbs' phenomenon for Euler means*, Acta Univ. Szeged. vol. 12 Part B (1950) pp. 107–111.

6. ———, *Gibbs' phenomenon for Hausdorff means*, Trans. Amer. Math. Soc vol. 69 (1950) pp. 440–456.

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