CLASSES OF MAXIMUM NUMBERS ASSOCIATED WITH TWO SYMMETRIC EQUATIONS IN N RECIPROCALS[1]

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1. Introduction. As in papers [1][2] and [2], we let $\sum_{i,j}(1/x)$ stand for the elementary symmetric function of the jth order of the i reciprocals $(1/x_p)$, $(p=1, \cdots, i; i>0)$ with

$$\sum_{i,j}(1/x) \equiv 0 \text{ when } i<j \text{ or } j<0,$$
$$\equiv 1 \text{ when } j=0$$

($\sum_{i,j}(x)$ having a similar meaning for the x_p themselves).

In §§1–4, we consider the equation

$$\sum_{n,r}(1/x) + L_{r+1}\sum_{n,r+1}(1/x) + \cdots + L_s\sum_{n,s}(1/x) = b/a, \tag{1}$$
$$a = (c+1)b - 1,$$

in which r, s, n are positive integers with $r<s\leqq n$; each L_p, $(p=r+1, \cdots, s)$, is a non-negative integer; and b, c are arbitrary positive integers.

In §5, we shall state results for another equation which follow from the procedure of §§1–4.

For convenience, we define

$$\phi_p(1/x) = \sum_{p,r}(1/x) + L_{r+1}\sum_{p,r+1}(1/x) + \cdots + L_s\sum_{p,s}(1/x), \tag{2}$$
$$(r \leqq p \leqq n).$$

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[1] Due to ingenuity of the referee, we have changed our method of reasoning about this material; a different use of relations (6) here has caused us to modify our transformation procedure. On account of this, we have replaced our former "E-solutions" by much more general "admissible solutions." Consequently, we have generalized the result in our first manuscript to Theorem 1 here, and we have done so relatively briefly. Furthermore, we have identified existentially every admissible solution of equation (1) that we would use in passing from a given admissible solution x of (1), x different from the Kellogg solution w in (5), to w.

The material of §4 here would not have occurred to us; it is entirely due to the referee, and it is largely in his wording. We are extremely grateful for his assistance, so much so that we offered to share the title of this paper with him.

[2] Numbers in square brackets refer to papers whose titles appear in the list of references at the end of this article.

Consequently, (1) is expressed by $\phi_n(1/x)=b/a$ with a as in (1).

Hence, using the notation

$$(3)\quad N_p(x) = \sum_{p,p-r+1}(x) + L_{r+1}\sum_{p,p-r}(x) + \cdots + L_s \sum_{p,p-s+1}(x), \qquad (p = r, \cdots, n-1),$$

and (2), we also write, for convenience,

$$(4)\quad \phi_p(1/x) = \frac{1}{x_1\cdots x_r} + \sum_{t=r}^{p-1}\frac{1}{x_{t+1}}\cdot\frac{N_t(x)}{x_1\cdots x_t} \qquad (r < p \leqq n).$$

By definition, the Kellogg solution [3] of (1) is obtained by taking the variables $x_1, x_2, \cdots, x_{n-1}$ in this order as small as possible (among positive integers). In obtaining this solution, say w, one would have $w_p=1$, $(p=1, \cdots, r-1)$, and $w_r=c+1$. Also since the term summed in (4) equals

$$\frac{[aN_t(x)+1]-1}{x_{t+1}(ax_1\cdots x_t)},$$

which reduces to $(ax_1\cdots x_t)^{-1}-(ax_1\cdots x_{t+1})^{-1}$ when $x_{t+1}=aN_t(x)+1$—the Kellogg choice for x_{t+1} after the first t x's have been selected by his method—one readily completes the Kellogg solution $x=w$, namely:

$$(5)\quad \begin{array}{ll} w_p = 1, \quad (p = 1, \cdots, r-1), & w_r = c+1, \\ w_{p+1} = aN_p(w)+1, \quad (p = r, \cdots, n-2), & w_n = aN_{n-1}(w). \end{array}$$

Heretofore, we have called any solution of (1) in which

(i) $x_1 \leqq x_2 \leqq \cdots \leqq x_n$;

(ii) the x_p, $(p=1, \cdots, n-1)$, are positive integers

an E-solution of (1), and we have considered such solutions only. However, the procedure to be used here reveals that we can obtain greater generality by merely requiring that the elements x_j in each solution x of (1) be real numbers $\geqq 1$, satisfying (i), and such that

$$(6)\quad \phi_p(1/x) \leqq b/a - 1/(ax_1\cdots x_p), \qquad (p = r, \cdots, n-1).$$

Henceforth such solutions will be called *admissible solutions*, and we shall confine our attention to these.

Reasoning used in [1] shows that when x is an E-solution of (1), (6) holds for each indicated value of p. Furthermore, one readily finds that when $x=w$, the equality sign applies in (6) for each value of p. Being an E-solution of (1), w is admissible.

Now let $P(x)\equiv P(x_1, x_2, \cdots, x_n)$ be any nonconstant, symmetric polynomial in the n elements x_p, $(p=1, \cdots, n)$, with no negative coefficient.

The result which we are to prove relative to (1) is as follows.

THEOREM 1. *If x is an admissible solution and $x \neq w$, then $x_n < w_n$ and $P(x) < P(w)$.*

2. **Lemmas that we use in proving Theorem 1.** The following lemma is essentially Lemma 1a of [1].

LEMMA 1. *Let $Q(1/x)$ stand for a symmetric polynomial in the u reciprocals $(1/x_p)$, $(p=1, \cdots, u; u>1)$, which is not a mere constant and contains at least one positive, and no negative, coefficient; with i and j equal to distinct positive integers each less than or equal to u, let x_i, x_j, α, β be positive numbers with $\alpha < x_i \leqq x_j$; and suppose that the expression which is obtained by replacing in $Q(1/x)$ the numbers x_i, x_j by $(x_i-\alpha)$, $(x_j+\beta)$, respectively, equals $Q(1/x)$; then*

$$(7) \quad x_i x_j \leqq (x_i - \alpha)(x_j + \beta), \quad x_i^h + x_j^h < (x_i - \alpha)^h + (x_j + \beta)^h,$$

where h is any positive integer. Furthermore, the equality sign holds in (7) if, and only if, $Q(1/x)$ is a polynomial in $(x_1 \cdots x_u)^{-1}$.

LEMMA 2. *If x is an admissible solution for which the equality sign applies in (6) for $p=k$, $(r \leqq k < n)$, then $x_{k+1} > x_k$ except when (1) is $x_1^{-1} + x_2^{-1} = 1$.*

PROOF. By hypothesis and our definition of $\phi_p(1/x)$, we have

$$\begin{aligned}(ax_1 \cdots x_k)^{-1} = b/a - \phi_k(1/x) &= \phi_n(1/x) - \phi_k(1/x) \\ &\geqq \phi_{k+1}(1/x) - \phi_k(1/x) \\ &\geqq x_{k+1}^{-1} \sum_{k,r-1} (1/x),\end{aligned}$$

so that

$$x_{k+1} \geqq a \sum_{k,k-r+1} (x) \geqq a x_k \geqq x_k$$

with the inequality sign holding throughout except when $n=k+1=2$ and $r=1=a$; that is, except in the special case indicated in the statement of Lemma 2.

TERMINOLOGY. Henceforth, if for an admissible solution the sign $<$ holds in (6) for $p=k$, $(r \leqq k < n)$, to reduce the kth element broadly until the equality sign holds in (6) for $p=k$ will mean that:

(1°) we reduce the kth element only until the equality is reached if this does not require the kth element to be less than the $(k-1)$th;

(2°) if the kth element is equal to one or more elements with smaller subscript, or becomes so while being reduced before the equality for $p=k$ in (6) is reached, then each such element is to be kept equal to the kth in the rest of its reduction.

LEMMA 3. *If x is an admissible solution for which the inequality sign holds in* (6) *for $p=m$, $(r\leqq m\leqq n-1)$, and if we reduce the mth element of x broadly until the equality sign holds in* (6) *for $p=m$, the new set $(X_1, \cdots, X_m)$ obtained is such that*

$$X_1 \leqq X_2 \leqq \cdots \leqq X_m; \tag{8}$$

$$\phi_p(1/X) \leqq b/a - (aX_1 \cdots X_p)^{-1}, \qquad (r \leqq p \leqq m). \tag{9}$$

PROOF. By (1°) and (2°) above, reducing the mth element broadly until the equality sign holds in (9) for $p=m$ yields new elements X_u, $(u=1, \cdots, m)$, which satisfy (8).

Relative to (9), by hypothesis the equality sign holds in it when $p=m$; so if there is just one case in (9), namely $r=m$, then (9) is true. Suppose $r<m$. Then reducing x_m only would not change the value of $\phi_p(1/x)$, $(r\leqq p<m)$. Also if at some stage of our broad reduction of the mth element, the numbers x_p, $(p=1, \cdots, m)$ are x_p' with $x_m'=x_{m-1}'$, and if an admissible solution x' is completed by using x' instead of x in equations (11) below, then Lemma 2 guarantees that we could not have

$$\phi_{m-1}(1/x') = b/a - (ax_1' \cdots x_{m-1}')^{-1}:$$

so, in the reduction while $x_m'=x_{m-1}'$, we would have $\phi_{m-1}(1/x')<b/a-(ax_1' \cdots x_{m-1}')^{-1}$. Similarly, we could not have simultaneously

$$x_m' = x_p', \quad \phi_p(1/x') = b/a - (ax_1' \cdots x_p')^{-1}, \qquad (r \leqq p < m-1),$$

when there is at least one value of p satisfying the indicated conditions. Reasoning similarly about cases in which $x_u' = x_v'$ where u and v are distinct values of p, $(r\leqq p\leqq m-1)$, completes the proof that the set $X_1, \cdots, X_m$ which is obtained from $x_1, \cdots, x_m$ by decreasing x_m broadly satisfies (9).

Reasoning used in §1 about the term summed in (4) yields the following lemma.

LEMMA 4. *If x is an admissible solution for which the equality sign holds in* (6) *when $p=m$, $(r\leqq m\leqq n-1)$, then the equations*

$$\phi_p(1/x) = b/a - (ax_1 \cdots x_p)^{-1}, \qquad (p = m+1, \cdots, n-1), \tag{10}$$

hold if, and only if, the x's with subscript larger than m are as follows:

(11) $x_p = aN_{p-1}(x) + 1, \quad (p = m + 1, \cdots, n - 1), \quad x_n = aN_{n-1}(x),$

where for $m=n-1$ *all equations in* (10) *are to be discarded, and all in* (11) *except its last equation.*

3. **Proof of Theorem 1.** If x is an admissible solution for which the *inequality* sign holds in (6) for $p=n-1$, keeping the left member of (1) equal to b/a, we decrease x_{n-1} broadly until the equality sign holds in (6) for $p=n-1$ and we increase x_n accordingly. The new set X obtained satisfies the case $m=n-1$ of (8) and (9) with equality in the latter for $p=n-1$: and $X_n=aN_{n-1}(X)>x_n$. Obviously X is admissible. Furthermore, in transforming from x to X, every element that was decreased was no larger than the single element x_n which was increased. Consequently, by Lemma 1, $P(X)>P(x)$.

Next, if x is an admissible solution of (1) for which the inequality sign holds in (6) for $p=k$, $(r\leqq k<n-1)$, but not for p equal to any one of the numbers $k+1, \cdots, n-1$ (where this set is now not vacuous), so that, in particular for $p=k+1$,

$$\phi_{k+1}(1/x) + (ax_1 \cdots x_{k+1})^{-1} = b/a,$$

then maintaining this equality, we decrease x_k broadly until the equality sign holds in (6) for $p=k$, and we increase x_{k+1} accordingly. The first $(k+1)$ elements of the new set X satisfy the case $k+1=m$ of (8) and (9) with equality in (9) for $p=k+1$. By Lemma 1, the product and the sum of the hth powers of the $k+1$ elements $X_1, \cdots, X_{k+1}$, h any positive integer, exceed the product and sum of the hth powers, respectively, of the corresponding elements $x_1, \cdots, x_{k+1}$. (We let X_p in X correspond to x_p in x.) Furthermore, it is obvious that we can complete an admissible solution X of (1) by using X in the place of x in those cases of (11) which define the X_j $(j=k+2, \cdots, n)$; also, from the nature of our transformation and Lemma 1, we know that $N_p(X)>N_p(x)$ for $p=k+1, \cdots, n-1$. Consequently

(12) $$X_n > x_n; \qquad P(X) > (Px), \quad \text{(Lemma 1)}.$$

The two types of argument used in the last two paragraphs above guarantee that if x is an admissible solution for which the inequality sign holds in one or more cases of (6), we can transform x into another admissible solution X for which the equality sign holds in (6) for every p. Furthermore, such argument shows that (12) holds. Consequently, if $r=1$, so that X is necessarily w, we are through.

Next, suppose that $r>1$ and that x, $(x\neq w)$, is an admissible solution for which the equality sign holds in (6) for $p=r, \cdots, n-1$; now necessarily $x_{r-1}>1$. Then, keeping

$$\phi_r(1/x) + (ax_1 \cdots x_r)^{-1} = b/a,$$

we decrease to 1 every x_p, $(p=1, \cdots, r-1)$ that exceeds 1, and we increase x_r accordingly. The product of the first r elements is left unchanged $(=c+1)$, but, by Lemma 1, the sum of the hth powers of these elements is increased. Consequently, when we complete our admissible solution X by writing [cf. (11)]

$$X_p = aN_{p-1}(X) + 1, \quad (p = r + 1, \cdots, n - 1), \quad X_n = aN_{n-1}(X),$$

no one of these elements will be less than its correspondent in x, and, since $n>r$, we shall surely have $X_n > x_n$ and $P(X) > P(x)$. Since X is now necessarily w, our proof of Theorem 1 is complete.

4. Another proof of Theorem 1.

LEMMA 5. *The admissible solutions x—considered as points in the space of the variables $x_1, \cdots, x_n$ under our specified conditions (of admissibility)—form a closed, bounded region of n-space.*

PROOF. Suppose false. Then there exists a sequence of admissible solutions $x^{(1)}, x^{(2)}, \cdots$, such that $x_n^{(v)}$ tends to infinity as v does so. Since $0 < b/a = \phi_n(1/x)$, we see that $x_n^{(v)}$ does not tend to infinity for some fixed N, $1 \leqq N < n$. Without loss of generality, suppose N maximal. Then we have

$$\lim_{v\to\infty} \phi_N(1/x^{(v)}) = \lim_{v\to\infty} \phi_n(1/x^{(v)}) = b/a.$$

Hence $\lim_{v\to\infty} (ax_1^{(v)} \cdots x_n^{(v)})^{-1} = 0$, which is a contradiction since

$$1 \leqq x_1^{(v)} \leqq \cdots \leqq x_N^{(v)}.$$

Now to prove Theorem 1, it suffices to prove this: if x is an admissible solution $\neq w$, there exists another admissible solution y of (1) such that

$$y_n > x_n, \qquad P(y) > P(x).$$

By reference to §3, we see that the first X into which our procedure carried any admissible solution x other than w can be taken as y; and the proof is complete.

5. Another equation and our result for it.

Consider the equation

$$(13) \qquad L_r \sum_{n,r} (1/x) + L_{r+1} \sum_{n,r+1} (1/x) + \cdots + L_s \sum_{n,s} (1/x) = 1,$$

in which L_r is a positive integer; each L_i, $(i=r+1, \cdots, s)$, is a nonnegative integer; and r, s, n are positive integers with $r<s\leqq n$.

Equation (13) contains equations which are not included in (1), and our procedure of §§1–3 applies to (13). After a few definitions, we shall state our result for (13).

To obtain our new definitions of $\phi_p(1/x)$ and $N_p(x)$ from (2) and (3), respectively, merely multiply the first term in the right member of (2) and of (3) by L_r. Also we define any solution x of (13) to be admissible if each $x_i \geqq 1$, $(i=1, \cdots, n)$, $x_1 \leqq x_2 \leqq \cdots \leqq x_n$, and the case $a=b=1$ of (6) holds.

By procedure heretofore described, one finds the Kellogg solution of (13) to be $x=W$, where

$$\begin{aligned} W_p &= 1, \ (p = 1, \cdots, r-1), & W_r &= L_r + 1, \\ W_{p+1} &= N_p(W) + 1, \ (p = r, \cdots, n-2), & W_n &= N_{n-1}(W). \end{aligned} \tag{14}$$

THEOREM 2. *If x is an admissible solution of* (13) *other than W, of* (14), *then*

$$W_n > x_n, \qquad P(W) > P(x).$$

REFERENCES

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