NOTE ON SUMS OF FOUR AND SIX SQUARES

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1. Bailey [1] showed that Ramanujan's identity

$$\sum_{m=0}^{\infty} p(5m+4) = 5 \prod_{n=1}^{\infty} \frac{(1-x^{5n})^5}{(1-x^n)^6}$$

can be derived from the identity

$$\begin{aligned}(1)\qquad \sum_{-\infty}^{\infty} \left\{ \frac{xq^n}{(1-xq^n)^2} - \frac{yq^n}{(1-yq^n)^2} \right\} &= \frac{(x-y)(1-xy)}{(1-x)^2(1-y)^2} \\ &\cdot \prod_{1}^{\infty} \frac{(1-xyq^n)(1-x^{-1}y^{-1}q^n)(1-xy^{-1}q^n)(1-x^{-1}yq^n)(1-q^n)^4}{(1-xq^n)^2(1-x^{-1}q^n)^2(1-yq^n)^2(1-y^{-1}q^n)^2}\end{aligned}$$

which is equivalent to the familiar formula

$$\wp(u) - \wp(v) = -\frac{\sigma(u+v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)}.$$

Similarly the formula

$$\begin{aligned}(2)\qquad 1 + a^{-1}\frac{(1-a)^3}{1+a}\sum_{1}^{\infty} \frac{n^2q^{2n}}{1-q^{2n}}(a^n - a^{-n}) \\ = \prod_{1}^{\infty} \frac{(1-q^{2n}a^2)(1-q^{2n}a^{-2})(1-q^{2n})^6}{(1-q^{2n}a)^4(1-q^{2n}a^{-1})^4},\end{aligned}$$

which is equivalent to

$$\wp'(u) = -\sigma(2u)/\sigma^4(u),$$

can be used to prove various results involving partition functions. Dobbie [3] recently constructed simple direct proofs of (1) and (2) that require no knowledge of elliptic functions; incidentally (2) can be derived from (1) by dividing by $x-y$ and then letting $y \to x$.

The writer [2] showed that by means of (2) one can give a very concise proof of the familiar formula for the number of representations of an integer as a sum of eight squares or of eight odd squares. In the present note we obtain the formulas for four and six squares in a similar manner (see for example [6, p. 307]).

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2. We recall the formulas (see for example [**5**, p. 282])

$$\theta_0(q) = \sum_{-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{1}^{\infty} \frac{(1-q^n)^2}{1-q^{2n}}, \tag{3}$$

$$\theta_2(q) = 2\sum_{1}^{\infty} q^{(2n-1)^2/4} = 2q^{1/4}\prod_{1}^{\infty} \frac{(1-q^{4n})^2}{1-q^{2n}}, \tag{4}$$

$$\theta_3(q) = \theta_0(-q). \tag{5}$$

It follows from (3) and (4) that

$$\theta_0(q)\theta_3(q) = \theta_0^2(q^2), \qquad \theta_2^2(q) = 2\theta_2(q^2)\theta_3(q^2). \tag{6}$$

For the case of six squares we shall in addition require

$$\theta_3^4(q) = \theta_0^4(q) + \theta_2^4(q), \tag{7}$$

which incidentally is proved in §3 below.

We define $r_k(n)$, $r_k'(n)$ by means of

$$\theta_3^k(q) = \sum_{n=0}^{\infty} r_k(n) q^n, \qquad \theta_2^k(q^4) = \sum_{n=1}^{\infty} r_k'(n) q^n. \tag{8}$$

3. In (1) replace q by q^2 and then put $y=-x=q$. The left hand side of (1) becomes

$$\sum_{n=-\infty}^{\infty} \left\{ \frac{q^{2n+1}}{(1-q^{2n+1})^2} + \frac{q^{2n+1}}{(1+q^{2n+1})^2} \right\} = 2\sum_{n=0}^{\infty} \left\{ \frac{q^{2n+1}}{(1-q^{2n+1})^2} + \frac{q^{2n+1}}{(1+q^{2n+1})^2} \right\}.$$

The right hand side of (1) becomes

$$4q\prod_{n=1}^{\infty} \frac{(1-q^{4n})^4}{(1-q^{4n-2})^4} = 4q\prod_{n=1}^{\infty} \frac{(1-q^{4n})^8}{(1-q^{2n})^4} = \frac{1}{4}\theta_2^4(q).$$

Hence we have the identity

$$\begin{aligned} \sum_{n=0}^{\infty} r_4'(8n+4) q^{2n+1} = \theta_2^4(q) &= 8\sum_{n=0}^{\infty} \left\{ \frac{q^{2n+1}}{(1-q^{2n+1})^2} + \frac{q^{2n+1}}{(1-q^{2n+1})^2} \right\} \\ &= 16\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} (2r+1) q^{(2n+1)(2r+1)}, \end{aligned} \tag{9}$$

which is equivalent to the known results on sums of four odd squares.

In (1) let us now put $x=i$ and $y=-i$. The left hand side of (1) becomes

$$\sum_{n=-\infty}^{\infty}\left\{\frac{iq^n}{(1-iq^n)^2}+\frac{q^n}{(1+q^n)^2}\right\}$$

$$= -\frac{1}{4}+\sum_{n=1}^{\infty}\left\{\frac{iq^n}{(1-iq^n)^2}-\frac{iq^n}{(1+iq^n)^2}+\frac{2q^n}{(1+q^n)^2}\right\}$$

$$= -\frac{1}{4}+\sum_{n=1}^{\infty}\left\{-\frac{4q^{2n}}{(1+q^{2n})^2}+\frac{2q^n}{(1+q^n)^2}\right\}$$

$$= -\frac{1}{4}-2\sum_{n=1}^{\infty}(-1)^n\frac{q^n}{(1+q^n)^2}.$$

The right hand side of (1) becomes

$$-\frac{1}{4}\prod_{n=1}^{\infty}\frac{(1+iq^n)^2(1-iq^n)^2(1-q^n)^4}{(1-iq^n)^2(1+iq^n)^2(1+q^n)^4}$$

$$= -\frac{1}{4}\prod_{n=1}^{\infty}\frac{(1-q^n)^8}{(1-q^{2n})^4} = -\frac{1}{4}\theta_0^4(q).$$

Hence we have the identities

$$\theta_0^4(q) = 1+8\sum_{n=1}^{\infty}\frac{(-1)^nq^n}{(1+q^n)^2} = 1+8\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+r+1}rq^{nr}, \tag{10}$$

$$\begin{aligned}\sum_{n=0}^{\infty} r_4(n)q^n = \theta_3^4(q) = \theta_0^{(4)}(-q) &= 1+8\sum_{n=1}^{\infty}\frac{q^n}{(1+(-q)^n)^2}\\ &= 1+8\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{(n-1)(r-1)}rq^{nr},\end{aligned} \tag{11}$$

the latter of which is equivalent to the known results on sums of four squares.

From (9), (10) and (11) we see that

$$\theta_3^4(q) = \theta_0^4(q)+\theta_2^4(q),$$

which is (7) above. The writer is indebted to the referee for this observation.

4. Turning next to (2), we take $a=i$. This yields

$$1-4\sum_{m=1}^{\infty}\left(\frac{-4}{m}\right)\frac{m^2q^{2m}}{1-q^{2m}} = \prod_{1}^{\infty}\frac{(1-q^{2n})^4(1-q^{4n})^6}{(1-q^{8n})^4},$$

where $(-4/m)$ is the Jacobi symbol. The right member is equal to

$$\prod_{1}^{\infty}\frac{(1-q^{2n})^4(1-q^{4n})^8}{(1-q^{4n})^2(1-q^{8n})^4} = \theta_0^2(q^2)\theta_0^4(q^4) = \theta_0^4(q^2)\theta_3^2(q^2),$$

where we have used (3) and (6). We have therefore

$$1-4\sum_1^{\infty}\left(\frac{-4}{m}\right)\frac{m^2q^{2m}}{1-q^{2m}}=\theta_0^4(q^2)\theta_3^2(q^2). \tag{12}$$

Now take $a=qi$ in (2). We find that the right member becomes

$$2\frac{(1-qi)^4}{1+q^2}\prod_1^{\infty}\frac{(1-q^{2n})^8(1-q^{8n})^4}{(1-q^{4n})^6}=\frac{(1-qi)^3}{1+qi}\,\frac{\theta_0^4(q^2)\theta_2^2(q^2)}{2q};$$

the left member is equal to

$$\begin{aligned}1&+\frac{1}{q}\,\frac{(1-qi)^3}{1+qi}\sum_1^{\infty}\frac{n^2q^{2n}}{1-q^{2n}}\,i^{n-1}(q^n-(-1)^nq^{-n})\\ &=\frac{(1-qi)^3}{q(1+qi)}\left\{\frac{q(1+qi)}{(1-qi)^3}+\sum_1^{\infty}\frac{n^2q^{2n}}{1-q^{2n}}\,i^{n-1}(q^n-(-1)^nq^{-n})\right\}\\ &=2\,\frac{(1-qi)^3}{q(1+qi)}\sum_1^{\infty}\sum_1^{\infty}(-1)^{n-1}(2n-1)^2q^{(2r-1)(2n-1)},\end{aligned}$$

on expanding and combining. Thus we get

$$4\sum_1^{\infty}\sum_1^{\infty}(-1)^{n-1}(2n-1)^2q^{(2r-1)(2n-1)}=\theta_0^4(q^2)\theta_2^2(q^2). \tag{13}$$

If we divide by q, replace q^2 by $-q^2$, we find that (13) becomes

$$4\sum_1^{\infty}\sum_1^{\infty}(-1)^{r-1}(2n-1)^2q^{(2r-1)(2n-1)}=\theta_3^4(q^2)\theta_2^2(q^2). \tag{14}$$

Again, if we take $a=q^{1/2}$ in (2) and then replace q by q^4, we get without much difficulty

$$1+q^{-2}\frac{(1-q^2)^3}{1+q^2}\sum_1^{\infty}\frac{n^2q^{6n}}{1+q^{4n}}=\frac{(1-q^2)^3}{1+q^2}\prod_1^{\infty}\frac{(1-q^{8n})^4(1-q^{4n})^8}{(1-q^{4n})^2(1-q^{2n})^4},$$

$$64q^2\frac{1+q^2}{(1-q^2)^3}+64\sum_1^{\infty}\frac{n^2q^{6n}}{1+q^{4n}}=\theta_2^2(q^2)\theta_2^4(q);$$

hence by the second of (6)

$$16q^2\frac{1+q^2}{(1-q^2)^3}+16\sum_1^{\infty}\frac{n^2q^{6n}}{1+q^{4n}}=\theta_3^2(q^2)\theta_2^4(q^2). \tag{15}$$

If we subtract (13) from (14) and use (7), it is evident that

$$(16)\quad \theta_2^6(q^2) = 4\sum_1^\infty\sum_1^\infty \{(-1)^{r-1} - (-1)^{s-1}\}(2s-1)^2 q^{(2r-1)(2s-1)}.$$

Define

$$E_2(n) = \sum_{d|n}\left(\frac{-4}{d}\right)d^2, \qquad E_2'(n) = \sum_{d\delta=n}\left(\frac{-4}{d}\right)\delta^2;$$

then the right member of (16) becomes

$$4\sum_{m \text{ odd}} q^m\{E_2'(m) - E_2(m)\}.$$

This evidently implies

$$(17)\qquad r_6'(2m) = 4\{E_2'(m) - E_2(m)\} \qquad (m \text{ odd}).$$

On the other hand, addition of (12) and (15) gives after some simplification

$$\theta_6^6(q^2) = 1 + 16\sum_1^\infty\sum_1^\infty\left(\frac{-4}{r}\right)n^2q^{2nr} - 4\sum_1^\infty\sum_1^\infty\left(\frac{-4}{n}\right)n^2q^{2nr},$$

which implies

$$(18)\qquad r_6(n) = 16E_2'(n) - 4E_2(n).$$

The formulas (17) and (18) are the well-known results of Jacobi on six squares; the notation is that of Glaisher [4].

We remark that (14) and (15) imply results on the number of representations in the forms

$$4(x_1^2 + x_2^2 + x_3^2 + x_4^2) + u_1^2 + u_2^2, \quad 4(x_1^2 + x_2^2) + u_1^2 + u_2^2 + u_3^2 + u_4^2,$$

where the u_i are odd, x_i arbitrary.

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