ON LAURENT OPERATORS ON l_p

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1. Let $f(\phi)$ be a complex-valued function of bounded variation on $-\pi \leq \phi \leq \pi$ and let $f(\phi)$ have the Fourier expansion

$$(1) \quad f(\phi) \sim \sum_{n=-\infty}^{\infty} f_n e^{in\phi}.$$

It will be supposed that $f(\phi)$ has the normalization

$$(2) \quad f(\phi) = f(\phi + 0) \text{ if } -\pi \leq \phi < \pi \text{ and } f(\pi) = f(\pi - 0).$$

The set of such functions form a Banach space with the norm

$$(3) \quad \|f\| = |f_0| + \int_{-\pi}^{\pi} |df|.$$

Associate with f the Laurent operator $y = Fx$, where $F = F(f)$, $x = (\dots, x_{-1}, x_0, x_1, \dots)$, $y = (\dots, y_{-1}, y_0, y_1, \dots)$,

$$(4) \quad y_n = \sum_{k=-\infty}^{\infty} f_{n-k} x_k,$$

$$(5) \quad x \in l_p, \text{ i.e., } \|x\|_p = \left(\sum_{n=-\infty}^{\infty} |x_n|^p \right)^{1/p} < \infty.$$

According to Stečkin [4], the operator F on l_p is bounded for every fixed $p > 1$, that is, (4) and (5) imply that y exists and is in l_p and that

$$(6) \quad \|y\|_p \leq \text{Const. } \|x\|_p.$$

Furthermore, the norm $\|F\|_p$ of F (that is, the least Const. satisfying (6)) is subject to an inequality of the type

$$(7) \quad \|F(f)\|_p \leq B_p \|f\|,$$

where B_p is an absolute constant, depending only on p .

In what follows, $p > 1$ is fixed and F is considered as an operation $x \rightarrow y = Fx$ of the Banach space l_p into itself. Correspondingly, the subscript p on the norms $\|\dots\|_p$ will be omitted.

Note that (1), (4), Parseval's relation and the boundedness of $F(f)$ show that

$$(8) \quad F(f - \lambda) = F(f) - \lambda I$$

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if I is identity operator, and that

$$(9) \quad F(f)F(g) = F(fg)$$

if f, g are two functions of bounded variation; cf. [1].

Krabbe has shown [1] that, when f is continuous, the spectrum $\sigma(f)$ of $F(f)$ is a connected subset of the range of $f(\phi)$ and [2] that, when f is absolutely continuous,

$$\sigma(f) = \text{range of } f(\phi).$$

In view of (8), this means that

$$F \text{ has a bounded inverse if and only if } f \neq 0;$$

furthermore, by (8),

$$(10) \quad F^{-1}(f) = F(1/f).$$

The object of this note is to prove analogues of these statements when f is an arbitrary function of bounded variation (which need not be absolutely continuous or even continuous).

(*) *Let $f(\phi)$ be a function of bounded variation on $-\pi \leq \phi \leq \pi$ normalized by (2). Then*

$$(11) \quad \sigma(f) = \text{closure of the range of } f;$$

that is,

$$(12) \quad F \text{ has a bounded inverse if and only if } |f| \geq \text{const.} > 0,$$

in which case (10) holds.

Needless to say, (11) does not hold unless the normalization (2) holds. Of course, f can always be normalized by changing its definition on an at most denumerable set of ϕ -values. This change does not affect the operator $F = F(f)$. It is clear that the closure of the range of f is the set of λ -values such that

$$(13) \quad \text{meas } \{ \phi \mid \lambda - \mu < f(\phi) < \lambda + \mu \} > 0 \text{ for every } \mu > 0.$$

In view of this fact, (*) is an analogue of a theorem of Toeplitz dealing with functions $f(\phi)$ of class L^2 and with $p=2$; cf., e.g., [3, pp. 171-175].

The proof of the main part of (*), that the closure of the range of f is contained in $\sigma(f)$, will be much simpler than the arguments in [1; 2]. It will not depend on topological properties of the space of functions f or on the algebra of the set of operators $F(f)$.

2. Proof of (*). It will first be shown that, if $\tau(f)$ denotes the closure of the range of f , then

$$(14) \quad \sigma(f) \subset r(f).$$

This is quite simple and the proof is the same as in the Toeplitz case. Suppose that $\lambda \notin r(f)$, it will be shown that $\lambda \notin \sigma(f)$. In view of (8), it can be supposed that $\lambda = 0$. The assumption $0 \notin r(f)$ means that $|f| \geq \text{const.} > 0$. Hence $1/f$ is of bounded variation. But the relation (9) shows that $F^{-1}(f)$ exists and satisfies (10). This proves (14).

In order to complete the proof of (*), it must be shown that

$$(15) \quad \sigma(f) \supset r(f).$$

Let $-\pi \leq \alpha < \pi$ and let $f(\alpha) = \lambda$. Since $\sigma(f)$ is a closed set, (15) will be proved if it is shown that $\lambda \in \sigma(f)$.

As above, there is no loss of generality in supposing $\lambda = 0$. Thus, if for some α in $[-\pi, \pi)$,

$$(16) \quad f(\alpha) = 0$$

it will be shown that $0 \in \sigma(f)$.

To this end, let $\mu > 0$ be a small number satisfying

$$(17) \quad -\pi \leq \alpha < \alpha + \mu < \pi.$$

Let the sequence x be the Fourier coefficients of the characteristic function $x(\phi)$ of the interval $(\alpha, \alpha + \mu)$, so that

$$2\pi x_n = \int_{\alpha}^{\alpha+\mu} e^{-in\phi} d\phi.$$

Thus, according as $n \neq 0$ or $n = 0$,

$$(18) \quad 2\pi x_n = ie^{-in\alpha}(e^{-in\mu} - 1)/n \quad \text{or} \quad 2\pi x_0 = \mu.$$

Clearly, $x \in l_p$.

By virtue of (4) and of Parseval's relation, the element $y = Fx$ of l_p is the sequence of Fourier constants of the function

$$(19) \quad y(\phi) = f(\phi)x(\phi).$$

Define $f_{\mu}(\phi)$ on $-\pi \leq \phi \leq \pi$ by the formulae

$$(20) \quad f_{\mu}(\phi) = f(\phi) \quad \text{or} \quad f_{\mu}(\phi) = 0,$$

according as ϕ is or is not on the interval $\alpha \leq \phi < \alpha + \mu$. It is clear that $f_{\mu}(\phi)$ is of bounded variation and satisfies the normalization (2). It is also clear from (16) and (3) that

$$(21) \quad \|f_{\mu}\| \leq 3T(\mu),$$

if $T(\mu)$ is the total variation of $f(\phi)$ on $\alpha \leq \phi \leq \alpha + \mu$.

Since $x(\phi) = 0$ if $\alpha < \phi < \alpha + \mu$ fails to hold, (19) implies that

$$(22) \quad y(\phi) = f_{\mu}(\phi)x(\phi)$$

and so, by Parseval's relation,

$$(23) \quad y = F(f)x = F(f_{\mu})x.$$

Hence, by (7) and (21),

$$(24) \quad \|y\| \leq \|F(f_{\mu})\| \cdot \|x\| \leq 3B_p T(\mu) \|x\|.$$

Finally, $f(\alpha+0)=f(\alpha)$ implies that

$$(25) \quad T(\mu) \rightarrow 0 \quad \text{as} \quad \mu \rightarrow 0.$$

Consequently,

$$(26) \quad \|y\|/\|x\| \rightarrow 0 \quad \text{as} \quad \mu \rightarrow 0.$$

This shows that $F^{-1}(f)$ does not exist (as a bounded operator) and, hence, $0 \in \sigma(f)$. This completes the proof of (*).

3. Toeplitz's results in the case $p=2$ suggest the question as to whether or not

$$(27) \quad \lambda \text{ is in the point spectrum of } F$$

if and only if

$$(28) \quad \text{meas} \{ \phi \mid f(\phi) = \lambda \mid > 0.$$

According to the considerations leading to (22), the statement (27) holds if the set $\{ \phi \mid f(\phi) = \lambda \}$ contains an interval or if (28) holds and $p \geq 2$. On the other hand, if (27) holds and $p \leq 2$, then (28) holds.

The question of the equivalence of (27) and (28) will be left open. It can be mentioned that (28) implies (27) if the following assertion is true: Let S be an arbitrary closed set on $[-\pi, \pi]$ with $\text{meas } S > 0$. Then there exists a function $x(\phi) \not\equiv 0$ on $[-\pi, \pi]$ which vanishes outside of S and for which the sequence x of Fourier constants is an element of l_p .

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