ON A GENERALIZATION OF THE NOTION OF H^* -ALGEBRA¹

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1. Introduction. An H^* -algebra of W. Ambrose [1] has the property that the orthogonal complement of an ideal is an ideal of the same kind. The present work is an attempt to characterize an H^* -algebra in terms of this property. For this purpose it is necessary to generalize the concept of H^* -algebra by introducing so-called two-sided H^* -algebras. We assume merely that there are two involutions in the algebra: right $x \rightarrow x^r$ and left $x \rightarrow x^l$ such that $(yx, z) = (y, zx^r)$ and $(xy, z) = (y, x^l z)$. It is possible to characterize two-sided H^* -algebra in terms of the above relation imposed on ideals by making some additional assumptions on the ideal annihilators. Since every simple two-sided H^* -algebra is an H^* -algebra with the same topology we have also found a new characterization of a proper H^* -algebra.

The notation is adopted essentially from [1; 6] and [3] but unlike Ambrose we do not require that ideals be closed. We shall make a distinction between a minimal ideal and a minimal closed ideal. Also it is understood that a proper ideal is not dense in whole algebra and that an idempotent is a nonzero element.

2. Complemented and right complemented algebra. First structure theorem.

DEFINITION 1. Let A be a Banach algebra which is a Hilbert space. We shall call A a right complemented algebra (r. c. algebra) if it has the property that the orthogonal complement of every right ideal is again a right ideal. Similarly we define a left complemented algebra (l. c. algebra). We shall call an algebra complemented if it is at the same time r. c. and l. c.

As an example of a right complemented algebra one can take a right H^* -algebra introduced by M. F. Smiley [6].

EXAMPLE 1. Let α be a positive norm-increasing bounded operator on a Hilbert space H and let A be the algebra of operators of the Hilbert Schmidt type on H. Then A is a right H^* -algebra (hence a r. c. algebra) in the scalar product $(a, b) = [\alpha a, b]$, where [,] denote a trace scalar product of the Hilbert Schmidt operators: [a, b]= tr (a^*b) .

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DEFINITION 2. A right complemented algebra A will be called proper if $r(A) = \{x \in A \mid Ax = (0)\} = (0)$.

LEMMA 1. The orthogonal complement I^p of a two-sided ideal I in a proper right complemented algebra A is a two-sided ideal and is identical with both the left and the right annihilator of I.

PROOF. We know already that I^p is a right ideal. Since $I^pI \subset I^p \cap I = (0)$, I^p is contained in l(I) which is a two-sided ideal. Hence it is sufficient to prove that $I^p = l(I)$. Consider $I_1 = I \cap l(I)$, which is also a two-sided ideal. It is readily verified that $I_1 \subset r(A) = (0)$. Thus $l(I) = I^p$. It follows at once that $I^p = r(I)$.

It is easy to see that a semi-simple r. c. algebra is proper and that every r. c. algebra is a direct sum of its radical and a semi-simple r. c. algebra. Also it is always true for a proper r. c. algebra A that l(A) = (0).

We shall say that an element $x \in A$ is left self-adjoint if (xy, z) = (y, xz) holds for every $y, z \in A$. An element e will be called a left projection if it is idempotent and left self-adjoint.

LEMMA 2. Let R be a proper closed regular right ideal in an r. c. algebra A and let e be its relative identity such that $e \in \mathbb{R}^{p}$. Then e is a left projection.

PROOF. Since e is a relative identity we have $ex - x \in R$ for every $x \in A$. It follows that eR = 0 and $eR^p = R^p$; if $x \in R$, then $ex \in R$ and, since $ex \in R^p$ also, we have ex = 0; if $x \in R^p$, then $ex - x \in R^p \cap R$ and hence ex - x = 0, ex = x. In particular $e^2 = e$. Also e is left self-adjoint for if we consider x, $y \in A$ and write $x = x_1 + x_2$, $y = y_1 + y_2$ with x_1 , $y_1 \in R^p$, x_2 , $y_2 \in R$, we have:

$$(ex, y) = (ex_1 + ex_2, y_1 + y_2) = (x_1, y_1 + y_2) = (x_1, y_1).$$

 $(x, ey) = (x_1 + x_2, ey_1 + ey_2) = (x_1 + x_2, y_1) = (x_1, y_1).$

LEMMA 3. Every semi-simple r. c. algebra A contains a left projection.

PROOF. Let x be an element in A which does not have a right quasiinverse. Let R be the closure of the regular right ideal $\{xy+y|y\in A\}$. Then -x is a relative identity of R. We write -x=e+u with $e\in R^p$, $u\in R$; then one can easily check that e also is a relative identity of R. Hence e is a left projection.

Now we define the double orthogonality and the primitivity of a projection as in [1]. Also as in [1] we show that every left projection can be expressed as a finite sum of doubly orthogonal primitive left projections, and the closed right ideal eA, where e is a projection, is a

minimal closed ideal if and only if e is primitive. From the first part of the last statement it follows that every semi-simple r. c. algebra contains a primitive left projection.

THEOREM 1. Every semi-simple r. c. algebra A is a direct sum of simple r. c. algebras each of which is a two-sided ideal in A.

PROOF. Let e be a primitive left projection in A and let I be the smallest closed two-sided ideal containing e. It is easy to see that I is a minimal closed two-sided ideal. Then I^p is also a semi-simple r. c. algebra. Using Zorn's lemma we complete the proof.

This is the first structure (Wedderburn) theorem for r. c. algebras.

3. Two-sided H^* -algebras.

DEFINITION 3. A Banach algebra A is called a *two-sided* H^* -algebra if A is a Hilbert space and if for every $a \in A$ there are elements a^i and a^r in A such that $(ab, c) = (b, a^i c)$ and $(ba, c) = (b, ca^r)$ hold for every $b, c \in A$.

THEOREM 2. Every proper right H^* -algebra A is a two-sided H^* -algebra.

PROOF. Let $x \in A$ and let N_1 be the linear space spanned by x. Let $M_1 = N_1^p$, $M_2 = M_1^r = \{y \in A \mid y^r \in M_1\}$ and $N_2 = M_2^p$. Then N_2 is onedimensional. Take $u \in N_2$ so that $(x, x) = (x^r, u)$ (note that $(x^r, u) = 0$ for all $u \in N_2$ is impossible). We shall show that $u = x^i$. Let $y, z \in A$; then $zy^r = \lambda x + v$ where λ is some complex number and $v \in M_1$. It follows that $yz^r = \bar{\lambda}x^r + v^r$ with $v^r \in M_2$. Then we have: $(xy, z) = (x, zy^r)$ $= (x, \lambda x + v) = (x, \lambda x) = \bar{\lambda}(x, x) = \bar{\lambda}(x^r, u) = (\bar{\lambda}x^r, u) = (\bar{\lambda}x^r + v^r, u)$ $= (yz^r, u) = (y, uz)$.

4. Well-complemented algebra. Second structure theorem. It turns out that in order to prove the second structure theorem for (right, left) complemented algebras it is necessary to introduce a new axiom. The algebras with the new axiom will be called well-complemented.

DEFINITION 4. A semi-simple r. c. algebra A will be called *right well-complemented* (r. w. c.) if every proper right (left) ideal in A has a nonzero left (right) annihilator. A c. algebra will be called *well-complemented* (w. c.) if it is r. w. c.

A two-sided H^* -algebra furnishes an example of a w. c. algebra.

Now we prove a series of lemmas which will be used to prove the second structure theorem.

LEMMA 4. Let L be a left ideal in a Banach algebra A such that every

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member of L has a right quasi-inverse. Then L is contained in the radical of A.

PROOF. The lemma follows from the fact that if xy has a right quasiinverse u then yx has also a right quasi-inverse v = -(yx+yux).

LEMMA 5. Every closed nonzero right ideal R in an r. w. c. algebra A contains a left projection.

PROOF. Since A is semi-simple $l(R^p)$ contains an element x which does not have a right quasi-inverse. Consider the closed regular right ideal R_1 = closure of $\{xy+y | y \in A\}$ of which -x is a relative identity. Since $xR^p = (0)$, we have $R^p \subset R_1$ and hence $R_1^p \subset R$. We write -x= e+u with $e \in R_1^p$, $u \in R_1$; then e is again a relative identity of R_1 . By Lemma 2 e is a left projection.

LEMMA 6. If e is a primitive idempotent in an r. w. c. algebra A, then the right ideal P = eA is a minimal right ideal.

PROOF. We know already that P is a minimal *closed* ideal. So it is sufficient to show that if R is an ideal dense in P then $e \in R$. In this case we can find $x \in R$ such that x - e has a right quasi-inverse y. Then xy+x-e=0 and hence $e \in R$.

Combining Lemma 6 with the technique used in [1] we prove:

LEMMA 7. Let $\{e_i\}$ be a family of primitive idempotents in a simple algebra A. Let $A_{ij} = e_i A e_j$. Then:

(i) Each A_{ii} is isomorphic to the complex field;

- (ii) Each A_{ij} is one-dimensional;
- (iii) $A_{ij}A_{jk} = A_{ik}$.

We shall say that an element x in a c. algebra A has a left adjoint if there exists an element $x^{l} \in A$ such that $(xy, a) = (y, x^{l}z)$ holds for all y, $z \in A$.

LEMMA 8. If e is a primitive left projection in an r. w. c. algebra A then every element in Ae has a left adjoint.

PROOF. Let $x \in Ae$. Consider the right ideal R = xA = xeA. We may assume that $ex \neq 0$ (otherwise we replace x by y = x + e and prove that y has a left adjoint). Then since $exe = \lambda e$ we have eR = eA from which it follows that R is closed. Hence R contains a left projection f = xz = xez, where z is some element in A. Then $fe \neq 0$ (otherwise $0 = ef = exez = \lambda ez$, since ef is left adjoint of fe) and hence $fe = xeze = \mu x$ or $x = 1/\mu fe$ from which follows that x has a left adjoint $x^i = 1/\overline{\mu}ef$.

Now we are in position to prove the second structure theorem.

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THEOREM 3. Every r. w. c. algebra is a two-sided H^* -algebra. In particular a simple r. w. c. algebra A is isomorphic to the algebra described in Example 1.

PROOF. We follow the technique used in [1] and [6]. Let $\{e_i\}$ be a maximal family of doubly orthogonal primitive left projections in A. Consider $R = \sum_{i \in J} e_i A$, where J is the set of indices in $\{e_i\}$. Then R is closed. If $R \neq \overline{A}$ then R^{p} contains a primitive left projection e. Then $(e_ie, e_ie) = (e, e_ie) = 0$, $e_ie = 0$ for every $i \in J$, i.e., e is doubly orthogonal to every e_i , which leads to a contradiction. Thus R=A. Consider $L = \sum_{i \in J} Ae_i$ and suppose $\overline{L} \neq A$. Then the right annihilator r(L) is a nonzero right ideal. This simply means that there is an element $x \in A$ such that $e_i x = 0$ for all e_i . Then for any $y \in e_i A$ we have $(e_iy, x) = (y, e_ix) = 0$, i.e., x is orthogonal to all e_iA , hence to whole A, which is a contradiction. Thus L is dense in A. Let us use the notation $A_{ij} = e_i A e_j$; then $L = \sum_{i,j} A_{ij}$. It follows from Lemma 8 that every element in A_{ij} has a left adjoint (in A_{ji}). So we choose the matrix units $e_{ij} \in A_{ij}$ such that $e_{ii} = e_i$ and $e_{ij}^l = e_{ji}$. We define the matrix (α_{ij}) by setting $\alpha_{ij} = (e_{ki}, e_{kj})$. It is easy to see that α_{ij} does not depend upon k and that the matrix (α_{ij}) is self-adjoint. Any two elements in L have the form $x = \sum_{i,j} x_{ij} e_{ij}$ and $y = \sum_{i,j} y_{ij} e_{ij}$, where x_{ij} and y_{ii} are suitable complex numbers, and the scalar product has the form:

$$(x, y) = \sum x_{ik}\alpha_{kj}\bar{y}_{ij} = \operatorname{tr}(x\alpha y^*),$$

where **x**, **y** and **\alpha** here stand for matrices (x_{ij}) , (y_{ij}) and (α_{ij}) respectively.

Now we shall show that (α_{ij}) represents a bounded operator on $L^2(J)$. For this purpose let us consider the conjugate-linear mapping $T: x \rightarrow x^l$ restricted to Ae_1 , where 1 is some fixed index in J. Since every element in Ae_1 has a left adjoint T is defined everywhere on Ae_1 ; the range of T is a subset of e_1A (in fact one can show that the range is entire e_1A). The graph of T is closed: if $\langle x_n, x_n^l \rangle \rightarrow \langle x, u \rangle$, then $x_n \rightarrow x$ and $x_n^l \rightarrow u$, and for every $y, z \in A(x_ny, z) \rightarrow (xy, z)$ and $(y, x_n^lz) \rightarrow (y, uz)$ from which it follows that $x^l = u$, and so $\langle x, u \rangle$ also belongs to the graph of T. From the closed graph theorem it follows that T is continuous. Thus there exists a positive number M such that

$$(*) \qquad (x^{l}, x^{l}) \leq M(x, x)$$

holds for all $x \in Ae_1$.

Now there is a natural 1-1 correspondence between elements of $L^2(J)$ and Ae_1 , in which a member x(i) of $L^2(J)$ corresponds to the element $x = \sum_{i \in J} x(i)e_{i1}$ and Ae_1 . If x(i) and y(i) are finite sequences

in $L^2(J)$ then $x^l = \sum_i \bar{x}(i)e_{1i}$, $y^l = \sum_i \bar{y}(i)e_{1i}$ and $(x^l, y^l) = \sum_{i,j} \bar{x}(i)\alpha_{ij}y(j)$. By the continuity of $T(x^l, y^l) = \sum_{i,j} \bar{x}(i)\alpha_{ij}y(j)$ holds for all x(i), y(i) in $L^2(J)$. From (*) we have then $\sum_{i,j} x(i)\alpha_{ij}\bar{x}(j) \le \alpha_{11}M\sum_i |x(i)|^2$. Completing the proof as in [6] we show that L is an algebra of the type described in Example 1. Since L is complete we have L = A. So A is a left and hence a two-sided H^* -algebra.

To conclude this section we construct a complemented algebra which is not well-complemented.

EXAMPLE 2. Let $\tilde{\alpha}$ be some positive norm-increasing unbounded operator with domain dense in some Hilbert space H. Let A be the set of all operators a on H such that $a\tilde{a}$ is an operator of the Hilbert Schmidt type. Then A is a complemented algebra in the scalar product $(a, b) = [a\tilde{\alpha}, b\tilde{\alpha}] = tr(a\tilde{\alpha}(b\tilde{\alpha})^*)$. It is easy to see that there are two dense subsets of elements in A, every element of one having the left adjoint $a^{l} = a^{*}$ and every element of the other having the right adjoint $a^r = \tilde{\alpha}^2 a^* \tilde{\alpha}^{-2}$ in A. It remains to show that A is not wellcomplemented. Let us denote by \tilde{A} the algebra of all operators of the Hilbert Schmidt type on H, and let e be some left projection in A. It is easy to show that the Hilbert space H can be realized as \tilde{Ae} and so that if $\tilde{x} \in H$ corresponds to $x \in \tilde{A}e$ and α is any operator such that $\alpha(x)$ is defined then $\alpha(\tilde{x}) = \alpha x$. Since $\tilde{\alpha}$ is unbounded there exists an $a \in \tilde{A}e = Ae$ such that $\tilde{\alpha}a$ is not of the Hilbert Schmidt type. This means that a does not have the left adjoint. Then from Lemma 8 it follows that A is not well-complemented.

5. A special realization of a well-complemented algebra.

EXAMPLE 3. Let λ be a norm-decreasing linear transformation from a Hilbert space H_2 onto a Hilbert space H_1 which has a bounded inverse transformation μ from H_1 onto H_2 . Let A be the set of all Hilbert Schmidt operators a from H_1 into H_2 . Let us define the multiplication by $a \circ b = a\lambda b$. Then A is a w. c. algebra in the scalar product (a, b) $= \text{tr} (ab^*)$. All the laws of an algebra are easily verified; $||ab|| \leq ||a|| ||b||$ follows from the fact that λ is norm-decreasing. The right and the left adjoint of an element $a \in A$ are defined by $a^r = \mu a^* \lambda^*$ and $a^l = \lambda^* a^* \mu$.

It turns out that every simple w. c. algebra A is of the above form. This is shown in the next theorem.

THEOREM 4. Every simple w. c. algebra A is of the form described in Example 3.

PROOF. Let $\{e_i\}$ and $\{f_k\}$ be maximal families of doubly orthogonal primitive left and right projections in A respectively. Then $A = \sum_{j,k} e_j A f_k$. We choose e_{ij} and f_{kl} as in Theorem 3, such that

 $e_{jj} = e_j, e_{1j}^l = e_{ji}, f_{kk} = f_k, f_{kl}^r = f_{lk}$. Let J = (j) and K = (k) be the sets of indices in $\{e_j\}$ and $\{f_k\}$ respectively; let $1 \in J$ and $2 \in K$ be some fixed indices. Choose $w_{12} \in e_1Af_2$ such that $||w_{12}|| = 1$ and let $w_{jk} = e_{j1}w_{12}f_{2k}$ for each $\langle j, k \rangle \in J \times K$. Then $w_{jk} \in e_jAf_k$ and $||w_{jk}|| = 1$. Since every e_jAf_k is one-dimensional, w_{jk} constitute an orthonormal base for A. Thus every $x \in A$ has the form $x = \sum_{j,k} x(j, k)w_{jk}$ and so the scalar product is of the form $(x, y) = \sum_{j,k} x(j, k)\bar{y}(j, k)$.

Now consider $w_{1k}w_{j2}$; since $w_{1k}w_{j2} \in e_1Af_2$ we have $w_{1k}w_{j2} = \lambda_{kj}w_{12}$ for some complex λ_{kj} . Multiplying both sides of the last equality with e_{i1} on the left and with f_{2l} on the right we get $w_{ik}w_{jl} = \lambda_{kj}w_{il}$. Thus the multiplication has the form $xy = \sum_{i,k} x(i,k)w_{ik} \sum_{j,l} y(j, l)w_{jl}$ $= \sum_{i,l} (\sum_{k,j} x(i, k)\lambda_{kj}y(j, l))w_{il}$.

It remains to show that the matrix λ_{kj} regarded as an operator from $H_2 = L^2(J)$ into $H_1 = L^2(K)$ is norm-decreasing and has an inverse. Since $w_{ik} \in e_i A$ we have $w_{ik} = \sum_{h} \tau_{kh} e_{ih}$ for some matrix $\{\tau_{kh}\}$ which can be shown to be independent of i; also $e_{ih} = \sum_{l} \mu_{hl} w_{il}$ where $\{\mu_{hl}\}$ is independent of i. It is easy to show that $\{\mu_{hl}\}$ is the inverse of $\{\tau_{kh}\}$. From the other hand $\lambda_{kj} w_{il} = w_{ik} w_{jl} = \sum_{h} \tau_{kh} e_{ih} w_{jl} = \tau_{kj} e_{ij} w_{jl}$ which simply means that $\lambda_{kj} = \tau_{kj}$ and that $\{\mu_{hl}\}$ is an inverse of $\{\lambda_{kj}\}$. In order to show that $\{\lambda_{kj}\}$ is norm-decreasing we consider $\alpha_{ij} = (e_{1i}, e_{1j}) = (\sum_{k} \mu_{ik} w_{1k}, \sum_{l} \mu_{jl} w_{1l}) = \sum_{k} \mu_{ik} \mu_{jk}$ which simply means that $\alpha = \mu \mu^*$. From the fact that α is positive and norm-increasing it is easy to derive that μ is also norm-increasing. The rest of the proof follows immediately.

Thus the algebra of Example 3 is essentially a most general w. c. algebra. All w. c. algebras are obtained by considering all possible direct sums of the algebras of the form of that described in Example 3.

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