ON INFINITELY DIFFERENTIABLE POSITIVE DEFINITE FUNCTIONS[1]

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1. Suppose that $f(x)$ is an infinitely differentiable positive definite function. That is to say

$$f(x) = \int_{-\infty}^{\infty} e^{itx} d\alpha(t), \tag{1}$$

where $d\alpha(t)$ is a bounded non-negative measure. Since $f(x)$ is infinitely differentiable, it is well known (cf. C.-G. Esseen [**4**, p. 24]) that

$$f^{(n)}(x) = \int_{-\infty}^{\infty} i^n t^n e^{itx} d\alpha(t).$$

Therefore, the sequence $\{(-i)^n f^{(n)}(0)\}_0^\infty$ represents a Hamburger moment sequence. Again, if $\{\xi_k\}_0^n$ is an arbitrary finite set of complex numbers and m is any non-negative integer, then

$$\begin{aligned}\left|\sum_{k=0}^{n} \xi_k (-i)^k f^{(k+m)}(x)\right|^2 &= \left|\int_{-\infty}^{\infty} t^m e^{itx} \sum_{k=0}^{n} \xi_k t^k d\alpha(t)\right|^2 \\ &\leqq \int_{-\infty}^{\infty} t^{2m} d\alpha(t) \int_{-\infty}^{\infty} \left|\sum_{k=0}^{n} \xi_k t^k\right|^2 d\alpha(t) \\ &= M_m \sum_{r=0}^{n} \sum_{s=0}^{n} \xi_r \bar{\xi}_s (-i)^{r+s} f^{(r+s)}(0),\end{aligned}$$

where $M_m = (-i)^{2m} f^{(2m)}(0)$.

It turns out that if we add to these two necessary conditions a third condition, namely that $\{(-i)^n f^{(n)}(0)\}$ is a *determined* Hamburger moment sequence,[2] then these three conditions are sufficient for an infinitely differentiable function to have the representation (1). However, even more is true. If $f(x)$ is defined and infinitely differentiable on some open interval containing the origin and satisfies the above conditions, then it has the representation (1); i.e. it can be extended to be a positive definite function. Since the Hamburger moment sequence is determined the extension is clearly unique (cf.

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[2] By this we mean that there exists a *unique* non-negative measure $d\alpha(t)$ such that $(-i)^n f^{(n)}(0) = \int_{-\infty}^{\infty} t^n d\alpha(t)$.

Esseen [4, pp. 24–25]). Moreover, we shall show that if the Hamburger sequence is not determined, the first two necessary conditions are not sufficient. On the other hand the fact that the Hamburger sequence $\{(-i)^n f^{(n)}(0)\}$ is determined is in general not a necessary condition.

The theorem we shall prove in this note is as follows:

THEOREM. *Let $f(x)$ be an infinitely differentiable function defined on the open interval $(-a, b)$ where $a, b>0$. If*

(a) $\{(-i)^k f^{(k)}(0)\}_0^\infty$ *is a determined Hamburger moment sequence and*

(b) *for every non-negative integer m there exists an $M_m>0$ such that for every $x\in(-a, b)$ and every finite set $\{\xi_k\}_0^n$ of complex numbers*

$$\left|\sum_{k=0}^n \xi_k(-i)^k f^{(k+m)}(x)\right|^2 \leq M_m \sum_{r=0}^n \sum_{s=0}^n \xi_r\bar{\xi}_s(-i)^{r+s} f^{(r+s)}(0),$$

then there exists a bounded non-negative measure $d\alpha(t)$ such that

$$f(x) = \int_{-\infty}^{\infty} e^{itx} d\alpha(t).$$

As a tool in the proof of this theorem we shall use the theory of operators in Hilbert space. This theorem was inspired by a recent result of A. P. Calderon and A. Devinatz [2; 3] when we noticed, that after some preliminary work, the same methods as used in [2] and [3] could be used to obtain our more general result.

2. In this section we shall construct the requisite tool which we shall use in the proof of our theorem, namely a Hilbert space. To do this we consider the class $\mathfrak{F}'$ of functions of the form

$$g(x) = \sum_{k=0}^n \xi_k(-i)^k f^{(k)}(x).$$

If $h(x)$ is another element of $\mathfrak{F}'$, namely

$$h(x) = \sum_{k=0}^m \eta_k(-i)^k f^{(k)}(x),$$

we shall construct an inner product in $\mathfrak{F}'$ by the formula

$$(g, h) = \sum_{r=0}^n \sum_{s=0}^m \xi_r\bar{\eta}_s(-i)^{r+s} f^{(r+s)}(0). \tag{2}$$

To show that this is a well defined function, suppose that g and h have different representations; i.e.

$$g(x) = \sum_0^n \xi_k' (-i)^k f^{(k)}(x), \qquad h(x) = \sum_0^{m'} \eta_k' (-i)^k f^{(k)}(x).$$

Then

$$\sum_{r=0}^{n'} \sum_{s=0}^{m'} \xi_r' \bar{\eta}_s' (-i)^{r+s} f^{(r+s)}(0) = \sum_{r=0}^{n'} \xi_r'(i)^r \bar{h}^{(r)}(0)$$

$$= \sum_{r=0}^{n'} \sum_{s=0}^{m} \xi_r' \bar{\eta}_s (-i)^{r+s} f^{(r+s)}(0) = \sum_{s=0}^{m} \bar{\eta}_s (-i)^s g^{(s)}(0)$$

$$= \sum_{r=0}^{m} \sum_{s=0}^{m} \xi_r \bar{\eta}_s (-i)^{r+s} f^{(r+s)}(0).$$

This shows that the bilinear function in (2) is well defined. Moreover, since $\{(-i)^n f^{(n)}(0)\}$ is a moment sequence, $(g, g) \geqq 0$ and by condition (b) of the theorem if $(g, g) = 0$, then $g(x) \equiv 0$. Conversely if $g(x) \equiv 0$, $(g, g) = 0$. Therefore, the bilinear function defined in (2) is an actual inner product on $\mathfrak{F}'$.

In general, $\mathfrak{F}'$ is not complete with respect to this norm. We shall show that it can be completed to a Hilbert space $\mathfrak{F}$ of functions on $(-a, b)$. Suppose then that $\{g_n\}_1^\infty$ is a Cauchy sequence in $\mathfrak{F}'$. That is to say $\|g_n - g_m\| \to 0$ as $n, m \to \infty$. By condition (b) of the theorem $|g_n(x) - g_m(x)|$ goes uniformly to zero as $n, m \to \infty$. Therefore, there exists a continuous function $g(x)$ defined on $(-a, b)$ such that $|g_n(x) - g(x)| \to 0$ as $n \to 0$. This extended class of functions, which we get as pointwise limits of Cauchy sequences in $\mathfrak{F}'$, we shall designate by $\mathfrak{F}$. It is clear that $\mathfrak{F}$ is a linear space over the complex number field. It remains to extend the inner product from $\mathfrak{F}'$ to $\mathfrak{F}$ so that $\mathfrak{F}$ becomes a Hilbert space. If $g, h \in \mathfrak{F}$, there exist Cauchy sequences $\{g_n\}$, $\{h_n\} \subset \mathfrak{F}'$ such that $g_n(x) \to g(x)$ and $h_n(x) \to h(x)$. We shall define

$$(g, h) = \lim_{n \to \infty} (g_n, h_n). \tag{3}$$

That this limit exists is clear since

$$|(g_n, h_n) - (g_m, h_m)| \leqq \|g_n - g_m\| \|h_n\| + \|g_m\| \|h_n - h_m\|.$$

The quantities $\|h_n\|$ and $\|g_m\|$ are uniformly bounded and therefore $\{(g_n, h_n)\}$ is a Cauchy sequence. We must show yet that (3) is a well defined function.

Suppose that $\{\tilde{g}_n\}$, $\{\tilde{h}_n\} \subset \mathfrak{F}'$ are Cauchy sequences such that $\tilde{g}_n(x) \to g(x)$ and $\tilde{h}_n(x) \to h(x)$ uniformly in $(-a, b)$. Condition (b) of the theorem tells us that for any m, $\tilde{g}_n^{(m)}(x) \to g^{(m)}(x)$ and $\tilde{h}_n^{(m)}(x) \to h^{(m)}(x)$ uniformly in $(-a, b)$ and therefore, in particular, $\tilde{g}_n^{(m)}(0)$

$\to g^{(m)}(0)$ and $\tilde{h}_n^{(m)}(0) \to h^{(m)}(0)$. Now, clearly

$$\lim_{n\to 0} (g_n, h_n) = \lim_{n\to\infty} \lim_{m\to\infty} (g_n, h_m) = \lim_{m\to\infty} \lim_{n\to\infty} (g_n, h_m).$$

Further,

$$\lim_{n,m\to\infty} (g_n, h_m) - \lim_{n,m\to\infty} (\tilde{g}_n, \tilde{h}_m)$$
$$= \lim_{m\to\infty} \lim_{n\to\infty} (g_n - \tilde{g}_n, h_m) + \lim_{n\to\infty} \lim_{m\to\infty} (\tilde{g}_n, h_m - \tilde{h}_m).$$

Suppose that

$$h_m(x) = \sum_k \xi_{k,m}(-i)^k f^{(k)}(x)$$

and

$$\tilde{g}_n(x) = \sum_k \eta_{k,n}(-i)^k f^{(k)}(x).$$

Then we have

$$\lim_{m\to\infty} \lim_{n\to\infty} (g_n - \tilde{g}_n, h_m) = \lim_{m\to\infty} \lim_{n\to\infty} \sum_k \xi_{k,m}(-i)^k [g_n^{(k)}(0) - \tilde{g}_n^{(k)}(0)] = 0$$

and similarly

$$\lim_{n\to\infty} \lim_{m\to\infty} (\tilde{g}_n, h_m - \tilde{h}_m) = \lim_{n\to\infty} \lim_{m\to\infty} \sum_k \eta_{k,n}(i)^k [\bar{h}_m^{(k)}(0) - \bar{\tilde{h}}_m^{(k)}(0)] = 0.$$

This shows that the function defined in (3) is indeed well defined.

To show that this bilinear function is an inner product we first note that for every g in $\mathfrak{F}$ there exists a Cauchy sequence $\{g_n\} \subset \mathfrak{F}'$ such that

$$(g, g) = \lim_{n\to\infty} (g_n, g_n) \geqq 0.$$

Again, suppose that for $g \in \mathfrak{F}$, $(g, g) = 0$. Suppose $\{g_n\} \subset \mathfrak{F}'$ is a Cauchy sequence such that $g_n(x) \to g(x)$ and $(g, g) = \lim (g_n, g_n) = 0$. Then since

$$|g_n(x)|^2 = \left| \sum_k \xi_{k,n}(-i)^k f^{(k)}(x) \right|^2 \leqq M_0 \|g_n\|^2$$

we have that $g_n(x) \to 0$, which shows that $g(x) \equiv 0$. On the other hand if $g(x) \equiv 0$ on $(-a, b)$ then $g \in \mathfrak{F}'$ and $(g, g) = 0$. Therefore, $\mathfrak{F}$ forms a linear space with an inner product. The proof of the fact that $\mathfrak{F}$ is complete uses standard arguments and we leave this to the reader.

An important fact that we shall need in the future, a fact used previously in connection with the space $\mathfrak{F}'$, is that for any $g \in \mathfrak{F}$

$$(-i)^n g^{(n)}(0) = (g(x), (-i)^n f^{(n)}(x)). \tag{4}$$

This can be easily proved by taking a Cauchy sequence $\{g_n(x)\} \subset \mathfrak{F}'$ such that $g_n(x) \to g(x)$, noting that (4) is true for every g_n in this sequence and then passing to the limit.

3. Now that we have constructed the Hilbert space $\mathfrak{F}$ we can proceed with the proof of our theorem. First we wish to set up a conjugation operator on $\mathfrak{F}$. Consider first an element $g \in \mathfrak{F}'$; i.e. $g(x) = \sum_0^n \xi_k (-i)^k f^{(k)}(x)$. Define

$$Jg(x) = \sum_0^n \bar{\xi}_k (-i)^k f^{(k)}(x).$$

Now, $J^2 g(x) = g(x)$ and if $h(x) = \sum_0^m \eta_k (-i)^k f^{(k)}(x)$, then

$$(Jg, Jh) = \sum_{r=0}^n \sum_{s=0}^m \bar{\xi}_r \eta_s (-i)^{r+s} f^{(r+s)}(0) = (h, g).$$

Since $\mathfrak{F}'$ is dense in $\mathfrak{F}$, J can be extended to all of $\mathfrak{F}$ and is a conjugation operator.

Define an operator D, with domain $\mathfrak{F}'$, by the relation

$$Dg(x) = -i dg(x)/dx.$$

In other words, if $g(x) = \sum_0^n \xi_k (-i)^k f^{(k)}(x)$, then

$$Dg(x) = \sum_0^n \xi_k (-i)^{k+1} f^{(k+1)}(x).$$

If $h(x) = \sum_0^m \eta_k (-i)^k f^{(k)}(x)$, then

$$(Dg, h) = \sum_{r=0}^n \sum_{s=0}^m \xi_r \bar{\eta}_s (-i)^{r+s+1} f^{(r+s+1)}(0) = (g, Dh).$$

Therefore, D is a symmetric operator. Further, since it clearly permutes with J, it has a self-adjoint extension. We shall show that the self-adjoint extension is unique and is therefore the closure of D.

Suppose H is any self-adjoint extension of D, and $dE(t)$ its canonical resolution of the identity. If $f_0(x) = f(x)$, then

$$(-i)^n f^{(n)}(0) = (H^n f_0, f_0) = \int_{-\infty}^{\infty} t^n d(E(t) f_0, f_0).$$

Since by hypothesis $\{(-i)^n f^{(n)}(0)\}$ is a uniquely determined Hamburger moment sequence, the measure $d(E(t)f_0, f_0)$ is uniquely determined. Suppose we let $f_n(x)=(-i)^n f^{(n)}(x)$. The linear manifold generated by this class of elements is dense in $\mathfrak{F}$. Now,

$$(E(\lambda)f_n, f_m) = (E(\lambda)H^{n+m}f_0, f_0) = \int_{-\infty}^{\lambda} t^{n+m} d(E(t)f_0, f_0).$$

Therefore, for any n and m, $(E(\lambda)f_n, f_m)$ is uniquely determined in the sense that if H_1 is another self-adjoint extension of D, $dE_1(t)$ its canonical spectral measure, then $(E_1(\lambda)f_n, f_m)=(E(\lambda)f_n, f_m)$. This means however that D has only one self-adjoint extension, namely its closure.

What we have just proved means that D^*, the adjoint of D, is self-adjoint and is the closure of D. Therefore g is in the domain of D^* if and only if there exists a sequence $\{g_n\}\subset\mathfrak{F}'$ such that $g_n\to g$ and $Dg_n\to D^*g$ in the strong topology of $\mathfrak{F}$. This implies uniform pointwise convergence and therefore,

$$D^*g(x) = -idg(x)/dx.$$

Let us consider the group of unitary operators

$$U_x = \int_{-\infty}^{\infty} e^{itx} dE(t),$$

where $dE(t)$ is the canonical spectral measure of D^*. Let $g\in\mathfrak{F}$ be such that

$$U_x g = \int_{-c}^{c} e^{itx} dE(t)g,$$

where c is a positive finite number. It is clear that any such element belongs to the domain of D^*. Let us expand e^{itx}, as a function of t, in its Taylor series about the origin. Since this Taylor series is uniformly convergent in any finite interval we have for every x

$$\begin{aligned} U_x g &= \sum_0^{\infty} \frac{x^n}{n!}\int_{-c}^{c} i^n t^n dE(t) g \\ &= \sum_0^{\infty} \frac{x^n}{n!} i^n D^{*n} g, \end{aligned}$$

where the convergence is in the strong topology of $\mathfrak{F}$. But since convergence in the strong topology implies pointwise convergence we have for every $y\in(-a, b)$

$$U_x g(y) = \sum_0^{\infty} \frac{g^{(n)}(y)}{n!} x^n.$$

Since this series has an infinite radius of convergence for every $y\in(-a, b)$ it is well known (cf. R. P. Boas [1]) that $g(y)$ is analytic. Therefore if $x+y\in(-a, b)$

$$U_x g(y) = g(y + x). \tag{5}$$

Since the class of elements for which (5) holds is dense in $\mathfrak{F}$, (5) must hold for every element of $\mathfrak{F}$. If again we set $f_0(x)=f(x)$ then by means of the relationship (4) we get

$$f(x) = U_x f(0) = (U_x f_0, f_0) = \int_{-\infty}^{\infty} e^{itx} d(E(t)f_0, f_0).$$

This completes the proof of the theorem.

4. In this section we shall show that there exist functions $f(x)$ such that $\{(-i)^n f^{(n)}(0)\}$ is an *undetermined* Hamburger moment sequence and which satisfy condition (b) of our theorem but which are not positive definite. This is essentially the same example as given in [3].

Let $\{\mu_n\}_0^\infty$ be any undetermined Hamburger moment sequence. Then there exist two different bounded non-negative measures $d\alpha_1(t)$ and $d\alpha_2(t)$ such that

$$\mu_n = \int_{-\infty}^{\infty} t^n d\alpha_1(t) = \int_{-\infty}^{\infty} t^n d\alpha_2(t).$$

Let

$$f_1(x) = \int_{-\infty}^{\infty} e^{itx} d\alpha_1(t),$$

$$f_2(x) = \int_{-\infty}^{\infty} e^{itx} d\alpha_2(t).$$

Further, let

$$f(x) = \begin{cases} f_1(x) & \text{for} \quad x \geqq 0, \\ f_2(x) & \text{for} \quad x \leqq 0. \end{cases}$$

Then, $(-i)^n f^{(n)}(0)=\mu_n$ and condition (b) of the theorem is clearly satisfied. However, $f(x)$ cannot be a positive definite function since a positive definite function must satisfy the relation

$$f(x) = \bar{f}(-x).$$

However, we would get in this case

$$f_1(x) = \bar{f}_1(-x) = \bar{f}_2(-x) = f_2(x)$$

which would mean $d\alpha_1 = d\alpha_2$.

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ON A SERIES OF RAINVILLE INVOLVING LEGENDRE POLYNOMIALS

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1. The object of this paper is to obtain some relations involving Legendre polynomials with the help of a series given by E. D. Rainville. The results are believed to be new.

2. We start with the series given by E. D. Rainville

$$P_n(\cos\alpha) = \left(\frac{\sin\alpha}{\sin\beta}\right)^n \sum_{k=0}^{n} c_{n,k}\left[\frac{\sin(\beta-\alpha)}{\sin\alpha}\right]^{n-k} P_k(\cos\beta). \tag{2.1}$$

Putting $\beta=2\alpha$ and $\cos 2\alpha = x$, we get

$$2^{n/2}(1+x)^{n/2}P_n\left(\left(\frac{1+x}{2}\right)^{1/2}\right) = \sum_{k=0}^{n} c_{n,k}P_k(x). \tag{2.2}$$

From (2.2) and the orthogonal property

$$\begin{aligned}\int_{-1}^{1}(1+x)^{n/2}P_r(x)P_n\left(\left(\frac{1+x}{2}\right)^{1/2}\right)dx &= \frac{c_{n,\gamma}}{2^{n/2-1}(2\gamma+1)}, \qquad 0\leqq\gamma\leqq n,\\ &= 0, \qquad r>n.\end{aligned} \tag{2.3}$$

Using (2.3) with Adams' expansion (*Modern analysis*, p. 331) for

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