THE GENERALITY OF LOCAL CLASS FIELD THEORY (GENERALIZED LOCAL CLASS FIELD THEORY V)[1]

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1. The theorems of local class field theory are known to hold for all fields which are complete under a discrete rank one valuation and whose residue class fields (1) have no inseparable extension and (2) have, in any algebraic closure, exactly one (necessarily cyclic) extension of degree n for every integer $n>0$. Since complete fields with any given residue class field can be constructed by formal power series or Witt vectors, we shall restrict ourselves to the set of fields satisfying (1) and (2), calling them *quasi Galois fields* (qGf).

O. F. G. Schilling [**3**] has constructed qGf of characteristic 0 by use of formal power series. But the only qGf's of prime characteristic mentioned up to now are the Galois fields and those infinite algebraic extensions of them whose "degree" has no infinite part. It is easy to see that these are the only absolutely algebraic qGf of prime characteristic. If these should be the only qGf of prime characteristic it would mean that generalized local class field theory was only a limiting case of the classical theory so that the new methods used to prove the existence theorem [**5**; **6**] could be replaced by something much simpler.

We prove here that this is not the case. For example, there exist qGf whose absolutely algebraic subfield is the algebraic closure of the Galois field of order p. In fact we prove a much more general result, namely:

THEOREM 1. *Every absolutely algebraic field of characteristic p is the absolutely algebraic subfield of some qGf of transcendence degree* 1.

2. Let k be any field, k^c an algebraic closure of k, and $\mathfrak{G}$ the group of all automorphisms of k^c/k. $\mathfrak{G}$ is a compact topological group. Let the ring $\bar{I}$ be the Cartesian product over all primes l of the rings of l-adic integers with the usual product topology. There is a natural, continuous mapping $(m, \sigma)\rightarrow\sigma^m$ of $\bar{I}\times\mathfrak{G}$ into $\mathfrak{G}$ and every element σ of $\mathfrak{G}$ has a "generalized period" in $\bar{I}$, namely a generator of the ideal of all $m\in\bar{I}$ for which $\sigma^m=1$. For all this see Artin [**1**, pp. 171–177]. If F is any field with $k\subset F\subset k^c$ then F is a qGf if and only if the Galois group of k^c/F is the closure of a cyclic group generated by an

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element of $\mathfrak{G}$ whose generalized period is 0 so we see: *k has an algebraic extension which is a qGf if and only if $\mathfrak{G}$ contains an element of generalized period* 0. I do not know any convenient necessary and sufficient condition that this should be so but the following sufficient condition is enough to prove Theorem 1:

LEMMA 1. *If for every n the normal extension K/k contains at least one subfield which is cyclic of degree n over k then K/k has an automorphism of generalized period* 0.

PROOF. We show first that for every prime l, the Galois group $\mathfrak{G}$ of K/k contains elements whose generalized period is divisible by l^∞, i.e. is divisible by l^n for every n. Consider the set of all cyclic extensions C/k of degree l with $C\subset K$.

CASE 1. There exists a C which, for every n, can be embedded in a C' which is contained in K and is cyclic of degree l^n over k. In this case if σ is any element of $\mathfrak{G}$ which induces a generating automorphism on C/k then the period of σ on C' is divisible by l^n. Hence the generalized period of σ is divisible by l^∞. Let S_l be the set of all such elements σ: it is a closed subset of $\mathfrak{G}$.

CASE 2. For every cyclic C/k of degree l there is an $n(C)$ such that C cannot be embedded in a cyclic C'/k of degree l^n if $n>n(C)$. Since k has cyclic extensions of degree l^n for every n, there must exist an infinite family C_i of cyclic extensions of degree l such that $n(C_i)$ approaches infinity as i approaches infinity and no C_i is contained in the composite of $C_1, C_2, \cdots, C_{i-1}$. From this and the compactness of $\mathfrak{G}$ it follows that the set S_l of all elements of $\mathfrak{G}$ which induce a generating automorphism on C_i/k for every i is nonempty. Again it is a closed subset of $\mathfrak{G}$ and all its elements have generalized period divisible by l^∞.

Since in either case the elements of S_l are defined by their effect on an abelian extension of exponent l, the family $\{S_l\}$, where l runs through all primes, has the finite intersection property. Any element of the intersection of all S_l has generalized period 0.

REMARK. Lemma 1 is true not only for Galois groups but for any topological group which is inverse limit of a system of finite groups: one need only substitute cyclic groups of characters for cyclic fields in the above proof.

PROOF OF THEOREM 1. Let $\bar{r}$ be the Galois field of order p and k_0 any algebraic extension of $\bar{r}$. Let $k=\bar{r}(t)$ where t is transcendental and let T be the maximal everywhere unramified extension of k. Let σ be an automorphism of k^c/k which induces a generating automorphism of T/k and let K be the subfield of the maximal abelian extension of k left fixed by σ. By class field theory it is easy to see that the Galois group of K/k is isomorphic to $u/\bar{r}\cdot$ where u is the group of k-idèles which are everywhere units. It is easy to construct closed subgroups h, with $\bar{r}\cdot\subset h\subset u$, such that u/h is cyclic of any given order; so by Lemma 1 there is an automorphism τ of k^c/k which induces on K an automorphism of generalized period 0. (It is easy to verify that Case 1 of Lemma 1 occurs when $l=p$ and Case 2 occurs for all $l\neq p$.) Choose τ so that it is the identity on T; this is possible because $T\cap K=k$. Now $T=\bar{r}^c k$ and the Galois groups of T/k and $\bar{r}^c/\bar{r}$ are isomorphic to $\bar{I}$. Hence there is an $m\in\bar{I}$ such that the fixed point field for σ^m in $\bar{r}^c$ is exactly k_0. Let F be the fixed point field for $\sigma^m\tau$ in k^c. The maximal absolutely algebraic subfield of F is k_0, and F is a qGf because $\sigma^m\tau$ has generalized period 0 on K, hence also on k^c.

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From Lemma 1 it follows easily that the rational field has algebraic extensions which are qGf. Also it is easy to prove that *any field k is contained in a* qGf: adjoin to k all roots of unity of order prime to its characteristic; then adjoin a transcendental element and apply multiplicative and additive Kummer theory.

3. Since Theorem 1 does not give an explicit construction for qGf, the following is also of interest.

THEOREM 2. *Let k be an absolutely algebraic field of characteristic p. Let S_∞, S_{fin} be the sets of primes l for which the degree of $k/\bar{r}$ is, or is not, divisible by l^∞, and assume that $p\in S_{\text{fin}}$ and that k contains primitive lth roots of unity for every $l\in S_\infty$. Let Γ be the additive group of all rationals with denominator prime to the elements of S_∞ and F the formal power series field $S(k, \Gamma, 1)$ of all formal sums $\sum a_\alpha u^\alpha(\{\alpha\}$ well ordered, $a_\alpha\in k)$ as defined in* [2, §4] *or* [4, p. 23]. *Then F is a qGf and k its maximal absolutely algebraic subfield. The fields k satisfying these assumptions are the only ones which can be absolutely algebraic subfields of qGf constructed in this way.*

PROOF. If $l\neq p$, $l\in S_\infty$, and k contains primitive lth roots of unity then it contains primitive l^nth roots of unity for every n. Every algebraic extension of F is gotten by adjoining absolutely algebraic elements and mth roots of u where m is a product of powers of primes in S_∞. If the assumption $p\in S_{\text{fin}}$ were dropped, then $S(k, \Gamma, 1)$ would have inseparable extensions; if the assumption about roots of unity were dropped, it would have nonabelian extensions.

Theorem 2 is an analogue for characteristic p of one of Schilling's

constructions in [3]. I am grateful to I. Kaplansky and O. F. G. Schilling for discussion and correspondence concerning Theorem 2.

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A NOTE ON COMPLETELY PRIMARY RINGS

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A completely primary ring will mean a commutative ring with identity in which the ideal of nilpotent elements, called the radical, is a maximal ideal. For a completely primary ring A with radical N, $\bar{A}$ will mean the quotient ring A/N. It has been shown by E. Snapper[1] that if A is a completely primary ring of characteristic zero then A contains a field F isomorphic with $\bar{A}$. The purpose of this note is to extend and, incidentally, give another proof of Snapper's result.

Theorem. *If A is a completely primary ring of characteristic zero and N its radical, then A contains a field F isomorphic with $\bar{A}=A/N$ such that each a in A can be uniquely written in the form $a=f+n$, where $f\in F$, $n\in N$.*

Proof. First note that $x\notin N$ implies that x has an inverse, x^{-1}. By Zorn's lemma A contains a maximal ring F whose intersection with N is 0. This ring F is a field, for otherwise the set F^* of all elements of the form ab^{-1}, $0\neq b$, $a\in F$, is a field containing F, whose intersection with N is 0, a contradiction. To prove the theorem it is sufficient to show that A is identical with the subset A^* of A consisting of all

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[1] E. Snapper, *Completely primary rings*, III *Imbedding and isomorphism theorems*, Ann. of Math. vol. 53 (1951) pp. 207–234. See Remark 4.1, p. 218.