ON UNIFORM DISTRIBUTION AND THE DENSITY OF SUM SETS

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1. For any lattice point $\mathfrak{a}=(a_1, \cdots, a_k)$ in the k-dimensional euclidean space R^k, let $\|\mathfrak{a}\|=\max_{\kappa=1,\cdots,k}|a_\kappa|$. If A is an infinite set of such lattice points, we define for $x>0$ the counting function $A(x)$ to be the number of elements $\mathfrak{a}\in A$ satisfying $\|\mathfrak{a}\|\leqq x$. Then various densities for such sets A can be introduced as generalizations of the well-known densities of sets of non-negative integers. In particular, we shall denote the lower limit, the upper limit, and, in the case of its existence, the limit, of the sequence $A(x)/(2x)^k$, as x tends to infinity, by $d_*(A)$, $d^*(A)$, and $d(A)$, respectively. Furthermore, if A is restricted to have elements with non-negative coordinates only, we shall consider the corresponding expressions of the sequence $A(x)/x^k$ and denote them by $D_*(A)$, $D^*(A)$, and $D(A)$. According to the terminology in the case $k=1$, we shall call these limits the lower and upper asymptotic densities and the natural density[1] of A, respectively.

The sum set $A+B$ of two sets A and B in R^k is, as usual, defined to be the set of all points $\mathfrak{a}+\mathfrak{b}$, $\mathfrak{a}\in A$, $\mathfrak{b}\in B$, obtained by vector addition.

By an interval $I\subseteq R^k$ we mean the cartesian product of any k open intervals of R^1. The unit cube, i.e. the set of all points $\mathfrak{x}=(x_1, \cdots, x_k)$ with $0\leqq x_\kappa<1$ $(\kappa=1, \cdots, k)$, is denoted by C^k.

For any real number x, let $[x]$ denote the greatest integer $\leqq x$ and let $\{x\}$ be the fractional part $x-[x]$. Then, for every point $\mathfrak{x}=(x_1, \cdots, x_k)$, let $\{\mathfrak{x}\}=(\{x_1\}, \cdots, \{x_k\})$, and for every set $M\subseteq R^k$, let $\{M\}$ be the set of all points $\{\mathfrak{x}\}$ with $\mathfrak{x}\in M$.

The Jordan content of a set $M\subseteq R^k$ is denoted by $\mu_k(M)$.

A sequence $\mathfrak{x}_1, \mathfrak{x}_2, \cdots$ is called uniformly distributed (mod 1) if it has the following property: Let, for any interval $I\subseteq C^k$, N_I be the set of indices i such that $\{\mathfrak{x}_i\}\in I$. Then N_I has a natural density $D(N_I)=\mu_k(I)$.

2. In the case $k=1$ an analogue of Mann's Theorem has been proved for lower asymptotic densities by M. Kneser [**3**]. In the absence of a similar theorem for $k>1$ the present paper aims at establishing the inequality of the $(\alpha+\beta)$-theorem for a certain class of sets

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[1] See, for instance, B. Volkmann [6].

of lattice points defined by means of uniformly distributed sequences.

Let $\lambda_1, \cdots, \lambda_k$ be fixed positive irrational numbers and let, for any lattice point $\mathfrak{a}=(a_1, \cdots, a_k)$, $\mathfrak{p}(\mathfrak{a})=(\lambda_1 a_1, \cdots, \lambda_k a_k)$. Furthermore, if M is an open subset of C^k, let A_M be the set of lattice points $\mathfrak{a}$ such that $\{\mathfrak{p}(\mathfrak{a})\}\in M$. If then $\overset{+}{A}_M$ denotes the set of those elements of A_M which have non-negative coordinates only, the following theorem holds:

THEOREM 1. *For any two open sets $M_1\subseteq C^k$ and $M_2\subseteq C^k$, the following densities exist and satisfy the inequalities*:

$$D(\overset{+}{A}_{\{M_1+M_2\}}) \geqq \min\,(1, D(\overset{+}{A}_{M_1}) + D(\overset{+}{A}_{M_2})) \tag{1}$$

and

$$d(A_{\{M_1+M_2\}}) \geqq \min\,(1, d(A_{M_1}) + d(A_{M_2})). \tag{2}$$

For the proof the following lemmas are needed:

LEMMA 1. *Under the conditions of the theorem,*

$$\overset{+}{A}_{\{M_1+M_2\}} \supseteq \overset{+}{A}_{M_1} + \overset{+}{A}_{M_2}. \tag{3}$$

PROOF. Let $\mathfrak{a}$ be an element of $\overset{+}{A}_{M_1}+\overset{+}{A}_{M_2}$. Then there are lattice points $\mathfrak{a}_1$ and $\mathfrak{a}_2$ with non-negative coordinates such that $\mathfrak{a}=\mathfrak{a}_1+\mathfrak{a}_2$, $\{\mathfrak{p}(\mathfrak{a}_1)\}\in M_1$, $\{\mathfrak{p}(\mathfrak{a}_2)\}\in M_2$. Hence $\{\mathfrak{p}(\mathfrak{a}_1+\mathfrak{a}_2)\}\in\{M_1+M_2\}$ and consequently, $\mathfrak{a}\in\overset{+}{A}_{\{M_1+M_2\}}$.

LEMMA 2. *Under the conditions of the theorem,*

$$A_{\{M_1+M_2\}} = A_{M_1} + A_{M_2}. \tag{4}$$

PROOF. As in Lemma 1, the relation

$$A_{\{M_1+M_2\}} \supseteq A_{M_1} + A_{M_2} \tag{5}$$

is obtained immediately. To prove the opposite inclusion, let $\mathfrak{a}\in A_{\{M_1+M_2\}}$. Then $\{\mathfrak{p}(\mathfrak{a})\}\in\{M_1+M_2\}$, i.e.

$$\{\mathfrak{p}(\mathfrak{a})\} \equiv \mathfrak{m}_1 + \mathfrak{m}_2 \pmod 1, \quad \mathfrak{m}_1 \in M_1, \mathfrak{m}_2 \in M_2. \tag{6}$$

Since the sets M_1 and M_2 are open there exists an $\epsilon>0$ such that the interval

$$(m_{i1} - \epsilon, m_{i1} + \epsilon) \times (m_{i2} - \epsilon, m_{i2} + \epsilon) \times \cdots \times (m_{ik} - \epsilon, m_{ik} + \epsilon) \qquad (i = 1, 2)$$

is contained in M_i if $\mathfrak{m}_i=(m_{i1}, \cdots, m_{ik})$. By a theorem due to Hermann Weyl [7] each of the k sequences $\lambda_\kappa a$ $(a=0, 1, 2, \cdots)$ is uniformly distributed and therefore the sequences $\{\lambda_\kappa a\}$ are everywhere dense in the interval C^1. Thus there exists a lattice point $\mathfrak{a}_1=(a_{11}, \cdots, a_{1k})$ such that

$$m_{1\kappa} - \epsilon < \{\lambda_\kappa a_{1\kappa}\} < m_{1\kappa} + \epsilon \qquad (\kappa = 1, \cdots k) \tag{7}$$

and consequently, $\mathfrak{a}_1 \in A_{M_1}$. Letting $\mathfrak{a}_2=\mathfrak{a}-\mathfrak{a}_1$ one obtains

$$\{\mathfrak{p}(\mathfrak{a})\} - \{\mathfrak{p}(\mathfrak{a}_1)\} \equiv \{\mathfrak{p}(\mathfrak{a}_2)\} \pmod 1$$

and hence, if $\mathfrak{a}_2=(a_{21}, \cdots, a_{2k})$, (7) and (6) imply

$$\{\mathfrak{p}(\mathfrak{a}_2)\} \in (m_{21}-\epsilon, m_{2l}+\epsilon) \times \cdots \times (m_{2k}-\epsilon, m_{2k}+\epsilon),$$

therefore $\mathfrak{a}_2 \in A_{M_2}$ and $\mathfrak{a} \in A_{M_1}+A_{M_2}$. In view of (5), this establishes the lemma.

Lemma 3. *If the set of all lattice points in R^k is ordered as a sequence $\mathfrak{a}_1, \mathfrak{a}_2, \cdots$ in such a way that $\|\mathfrak{a}_m\| < \|\mathfrak{a}_n\|$ implies $m<n$, then the sequence $\mathfrak{p}(\mathfrak{a}_1), \mathfrak{p}(\mathfrak{a}_2), \cdots$ is uniformly distributed.*

Proof. From Weyl's theorem referred to above, it follows that each of the k sequences $\lambda_\kappa a$ $(a=0, \pm 1, \pm 2, \cdots)$ is uniformly distributed in the sense that, for any interval $I_\kappa \subseteq C^1$, the set A_{I_κ} has the density $d(A_{I_\kappa})=\mu_1(I_\kappa)$. By definition, the first $(2x+1)^k$ terms of the sequence $\mathfrak{a}_1, \mathfrak{a}_2, \cdots$ are exactly all the lattice points $\mathfrak{a}$ with $\|\mathfrak{a}\| \leqq x$. Therefore, if $I=I_1 \times I_2 \times \cdots \times I_k$ is an interval in C^k, then the counting function of the set A_I is

$$A_I(x) = A_{I_1}(x) A_{I_2}(x) \cdots A_{I_k}(x),$$

and thus the k asymptotic equations

$$A_{I_\kappa}(x) \simeq 2x \cdot \mu_1(I_\kappa) \qquad (\kappa = 1, \cdots, k)$$

imply

$$A_I(x) \simeq (2x)^k \prod_{\kappa=1}^{k} \mu_1(I_\kappa) = (2x)^k \mu_k(I)$$

and therefore $d(A_I)=\mu_k(I)$.

Since obviously $D(\overset{+}{A}_{I_\kappa})=d(A_{I_\kappa})$, one also obtains

$$D(\overset{+}{A}_I) = \mu_k(I).$$

Lemma 4. *For any open set $M \subseteq C^k$ there exist the densities*

$$d(A_M) = D(\overset{+}{A}_M) = \mu_k(M).$$

Proof. Let $\epsilon>0$, then there are sets R_ϵ and R^ϵ which are finite unions of intervals, such that $R_\epsilon \subseteq M \subseteq R^\epsilon$ and $\mu_k(R^\epsilon) - \mu_k(R_\epsilon) < \epsilon$. Then Lemma 3 implies that

$$\mu_k(R_\epsilon) \leqq d_*(A_M) \leqq d^*(A_M) \leqq \mu_k(R^\epsilon)$$

and therefore, since ϵ is arbitrary, $d(A_M) = \mu_k(M)$. The equation $D(A_M^+) = \mu_k(M)$ is obtained analogously.

LEMMA 5. *For any two open sets* $M_1 \subseteq C^k$, $M_2 \subseteq C^k$,

$$\mu_k(\{M_1 + M_2\}) \geqq \min(1, \mu_k(M_1) + \mu_k(M_2)). \tag{8}$$

PROOF. If all boundary points of C^k which the set $\{M_1 + M_2\}$ may contain are removed from it, the remaining set is obviously open. Therefore $\{M_1 + M_2\}$ has a content and (8) follows directly from A. M. Macbeath [**5**, Theorem 1].

Now Theorem 1 follows from Lemma 4, applied to the three sets M_1, M_2, and $\{M_1 + M_2\}$ and Lemma 5.

COROLLARY. *If* $M_1, \cdots, M_n$ *are open subsets of* C^1, *then*

$$A_{\{M_2 + \cdots + M_n\}} = A_{M_1} + \cdots + A_{M_n} \tag{9}$$

and

$$d(A_{\{M_1 + \cdots + M_n\}}) \geqq \min\left(1, \sum_{i=1}^{n} d(A_{M_i})\right). \tag{10}$$

PROOF. Follows from the theorem by induction.

3. In the case $k = 1$, Theorem 1 can be proved directly from Kneser's Theorem mentioned above, and consequently, the special case for linear, open sets of Macbeath's Theorem follows then as a corollary.

To establish this, we use the concept of a rational set of non-negative integers,[2] i.e. a set whose characteristic function with respect to the set of all non-negative integers is ultimately periodic. In this sense the set $A_{M_1}^+ + A_{M_2}^+$ is not rational whenever its lower asymptotic density is different from 1, for otherwise there would be some residue class P such that the intersection $P^+ \cap (A_{M_1}^+ + A_{M_2}^+)$ is empty or finite. If then P_1 and P_2 are any two residue classes such that $P_1^+ + P_2^+ = P^+$, it follows that at least one of the intersections $A_{M_1}^+ \cap P_1^+$ and $A_{M_2}^+ \cap P_2^+$, say, the first one, is empty or finite; otherwise $A_{M_1}^+ + A_{M_2}^+$ would contain infinitely many elements of P^+. But the sequence $\{\lambda_1 a\}$ with $a \in P_1^+$ is itself uniformly distributed[3] and must therefore, because of

[2] Cf. [1] and [6].
[3] Cf. [**7**].

$\mu_1(M_1)>0$, have infinitely many elements in M_1. This contradicts the assumption that $A^+_{M_1}+A^+_{M_2}$ is rational.

Kneser's Theorem implies that for any two sets A and B of non-negative integers whose sum set $A+B$ is not rational,

$$D_*(A+B) \geqq \min(1, D_*(A)+D_*(B)).$$

This proves (1) from which (2) can easily be obtained by decomposing the sets A_{M_1} and A_{M_2} into the subsets of their non-negative and of their negative elements.

4. The question may be raised what values the density $d(A_1+\cdots+A_n)$ can assume if $d(A_1), \cdots, d(A_n)$ are prescribed. As an answer to this question we prove the following

THEOREM 2.[4] *If $\alpha_1, \cdots, \alpha_n$, and γ are positive real numbers satisfying $\sum_{i=1}^n \alpha_i \leqq \gamma \leqq 1$, then there are sets $A_1, \cdots, A_n$ of lattice points in R^k such that*

$$d(A_i) = \alpha_i \ (i = 1, \cdots, n) \text{ and } d(A_1 + \cdots + A_n) = \gamma.$$

PROOF. In view of Lemmas 2 and 4 it suffices to show that there are open subsets $M_1, \cdots, M_n$ of C^k such that

$$(11) \quad \mu_k(M_i) = \alpha_i \ (i = 1, \cdots, n) \text{ and } \mu_k(\{M_1 + \cdots + M_n\}) = \gamma,$$

since the conditions of the theorem are then satisfied by the sets

$$A_i = A_{M_i}, \qquad A_1 + \cdots + A_n = A_{\{M_1+\cdots+M_n\}}.$$

Furthermore, we may restrict the proof to the case $k=1$; for, if $M_1, \cdots, M_n$ are subsets of C^1 satisfying (11), then the cartesian products $M_1\times\overline{C}^{k-1}, \cdots, M_n\times\overline{C}^{k-1}$ together with the set

$$\{(M_1\times\overline{C}^{k-1}) + \cdots + (M_n\times\overline{C}^{k-1})\} = \{M_1+\cdots+M_n\}\times\overline{C}^{k-1},$$

$\overline{C}^{k-1}$ being the interior of C^{k-1}, will satisfy (11) in the k-dimensional sense.

Such sets M_i can, for example, be constructed as follows: It may be assumed without loss of generality that $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n$. Then, for $i=1, \cdots, n-1$, let M_i be the open interval $(0, \alpha_i)$ and let $\sigma = \sum_{i=1}^{n-1}\alpha_i$. For the definition of M_n the following two cases are distinguished:

(a) If γ/σ is an integer q (hence $q \geqq 2$ since $\gamma-\sigma \geqq \alpha_n > 0$), let

$$M_n = \bigcup_{j=0}^{q-2}\left(j\sigma, j\sigma + \frac{\alpha_n}{q}\right) \cup \left((q-1)\sigma - \frac{\alpha_n}{q}, (q-1)\sigma\right).$$

[4] In the case of lower asymptotic densities of sets of integers a similar existence theorem was proved, by a different method, by L. P. Cheo [2] who used an idea of B. Lepson [4].

These q intervals are nonoverlapping as the distance of the "last" two is $\sigma-2\alpha_n/q=\sigma(1-2\alpha_n/\gamma)>0$ (for $\gamma\geqq\sum_{i=1}^n\alpha_i\geqq 2\alpha_n$), and otherwise the distance of any two adjacent intervals is $\sigma-\alpha_n/q>0$. Therefore, $\mu_1(M_n)=\alpha_n$ and, as is readily seen, $M_1+\cdots+M_{n-1}=(0,\sigma)$, consequently

$$\{M_1+\cdots+M_n\}=M_1+\cdots+M_n=\bigcup_{j=0}^{q-2}\left(j\sigma,(j+1)\sigma+\frac{\alpha_n}{q}\right)$$

$$\cup\left((q-1)\sigma-\frac{\alpha_n}{q},\ q\sigma\right)=(0,q\sigma)=(0,\gamma).$$

(b) If γ/σ is not an integer, a number $\epsilon>0$ is to be chosen such that

$$\epsilon<\min\left(\frac{\alpha_n}{\lfloor\gamma/\sigma\rfloor},\ \frac{1}{2}(\gamma-[\gamma/\sigma]\sigma)\right)$$

and M_n is defined as

$$M_n=(0,\alpha_n-\epsilon[\gamma/\sigma])\cup\left(\bigcup_{j=1}^{[\gamma/\sigma-1]}(j\sigma,j\sigma+\epsilon)\right)\cup(\gamma-\sigma-\epsilon,\gamma-\sigma).$$

Then as in case (a) the intervals of M_n are nonoverlapping as the "first" two of them obviously have a positive distance, the "last" two of them have the distance

$$(\gamma-\sigma-\epsilon)-\left(\left[\frac{\gamma}{\sigma}\right]-\sigma+\epsilon\right)=\gamma-\left[\frac{\gamma}{\sigma}\right]\sigma-2\epsilon>0$$

and otherwise the distance between any two neighboring intervals is

$$\sigma-\epsilon>\sigma-\frac{\alpha_n}{[\gamma/\sigma]}>\sigma\left(1-\frac{\alpha_n}{\gamma}\right)>0.$$

Thus

$$\mu_1(M_n)=\alpha_n-\epsilon\left[\frac{\gamma}{\sigma}\right]+\epsilon\cdot\left[\frac{\gamma}{\sigma}-1\right]+\epsilon=\alpha_n$$

and

$$\{M_1+\cdots+M_n\}=(0,\sigma)+M_n=(0,\gamma).$$

Added in proof (January 11, 1957). The inequalities of Theorem 1 may be expressed as

$$D(\overset{+}{A}_{M_1} + \overset{+}{A}_{M_2}) \geqq \min\,(1, D(\overset{+}{A}_{M_1}) + D(\overset{+}{A}_{M_2})), \tag{1*}$$

$$d(A_{M_1} + A_{M_2}) \geqq \min\,(1,\ d(A_{M_1}) + d(A_{M_2})) \tag{2*}$$

in view of Lemma 2 and the following

LEMMA 6. $D(\overset{+}{A}_{\{M_1+M_2\}}) = D(\overset{+}{A}_{M_1} + \overset{+}{A}_{M_2})$.

PROOF. Consider a covering of the set M_1 by a finite number of cubes C_i with $\mu_k(C_i) = \epsilon$, and let S be the set of points $\mathfrak{m}$ in $\{M_1 + M_2\}$ which have representations

$$\mathfrak{m} \equiv \mathfrak{m}_1 + \mathfrak{m}_2 \pmod 1, \qquad \mathfrak{m}_i \in M_i,$$

for at least all the points $\mathfrak{m}_1$ in one of the C_i's. Then the remaining set $R(\epsilon) = \{M_1 + M_2\} - S$ satisfies $\lim_{\epsilon \to 0} \mu_k(R(\epsilon)) = 0$. Due to uniform distribution, there is an $N(\epsilon)$ such that each C_i contains a point $\mathfrak{g}(\mathfrak{a}_1)$, $\mathfrak{a}_1 \in \overset{+}{A}_{M_1}$, with $\|\mathfrak{a}_1\| \leqq N(\epsilon)$. Thus, if $\mathfrak{a} = (a_1, \cdots, a_k) \in \overset{+}{A}_S$ and $a_\kappa \geqq N(\epsilon)$ for $\kappa = 1, \cdots, k$, then there is such a point $\mathfrak{a}_1$ for which $\{\mathfrak{g}(\mathfrak{a}) - \mathfrak{g}(\mathfrak{a}_1)\} \in M_2$ and, since all coordinates of $\mathfrak{a} - \mathfrak{a}_1$ are non-negative, $\mathfrak{a} - \mathfrak{a}_1 \in \overset{+}{A}_{M_2}$. Therefore, $\overset{+}{A}_S \subseteq \overset{+}{A}_{M_1} + \overset{+}{A}_{M_2}$, hence

$$D(\overset{+}{A}_S) = D(\overset{+}{A}_{\{M_1+M_2\}}) - \mu_k(R(\epsilon)) \subseteq D(\overset{+}{A}_{M_1} + \overset{+}{A}_{M_2}).$$

This proves the contention by virtue of Lemma 1.

Theorem 2 is also true for natural densities $D(A)$ of sets of lattice points with non-negative coordinates; in the proof reference has to be made to Lemma 6 instead of to Lemma 2.

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