ON THE DOMAIN OF REGULARITY OF GENERALIZED AXIALLY SYMMETRIC POTENTIALS

PETER HENRICI

Introduction. It is known [3; 7] that for $2\nu \neq 0, -1, -2, \dots$ every solution $u(x, y)$ of the partial differential equation¹

$$(1) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y} \frac{\partial u}{\partial y} = 0,$$

which depends analytically on the two real variables x and y in a domain containing a segment of the singular line $y=0$, is uniquely determined by its values on $y=0$. In this note we shall consider the following problem: *If the function $g(z) = u(z, 0)$ is continued to complex values of z , to what extent does its domain of regularity determine the domain of regularity of $u(x, y)$?*

A partial answer to this has been given by Erdélyi [1], who proved that if $g(z)$ is holomorphic in a y -convex domain² $\mathfrak{h}$, the function $u(x, y)$ is regular at all real points (x, y) for which $x+iy \in \mathfrak{h}$. The general results of I. N. Vekua in the theory of elliptic differential equations with analytic coefficients³ suggest the following wider (and perhaps more natural) statement:

THEOREM I. *Let $g(z)$ be holomorphic in a simply connected domain $\mathfrak{h}$, and let $2\nu \neq 0, -1, -2, \dots$. Then there exists a unique solution $u(x, y)$ of (1) satisfying $u(z, 0) = g(z)$ for $z \in \mathfrak{h}$; $u(x, y)$ is an analytic function of its arguments in the domain $D(\mathfrak{h})$ of all complex points (x, y) for which $x+iy \in \mathfrak{h}$, $x-iy \in \mathfrak{h}$.*

For $\text{Re } \nu > 0$ and convex domains $\mathfrak{h}$ Theorem I is contained in an otherwise more general result of ours.⁴ If x and y are restricted to real values and if $\mathfrak{h}$ is y -convex, the theorem reduces to Erdélyi's result.⁵

PROOF OF THEOREM I. Since uniqueness in the small, as established in [3], implies uniqueness in the large by analytic continuation, it suffices to show the existence of a function with the desired properties.

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¹ Frequently called equation of generalized axially symmetric potentials.

² By this is meant a domain which contains with the point $x+iy$ also the points $x+iyt$, $-1 \leq t \leq 1$.

³ See e.g. [6, §9].

⁴ See [2, Satz III,].

⁵ For a discussion of the role played by the exceptional values $2\nu = 0, -1, -2, \dots$ see [4].

We first assume $\operatorname{Re} \nu > 0$. If η is convex, the function

$$(2) \quad u(x, y) = \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu)\Gamma(1/2)} \int_{-1}^1 g(x + iyt)(1 - t^2)^{\nu-1} dt,$$

where the integration is along the real axis, meets the conditions of the theorem (see [2, proof of Satz III_ν]). In the nonconvex case the difficulty arises that no longer one and the same path of integration can be used for all points $P = (x, y) \in D(\eta)$. We therefore integrate along a curvilinear path which depends continuously on P . This path $C(P)$ may be constructed in the following manner: Let ϕ be a differentiable 1-1 mapping of η on a convex domain η' , and let $P' = (x', y')$ where $x' \pm iy' = \phi(x \pm iy)$. Let ζ_P denote the mapping of the t -plane on the T -plane given by $T = x + iyt$. This mapping depends analytically on the parameter P and so does its inverse, except for $y = 0$. We now put

$$C(P) = \begin{cases} \zeta_P^{-1} \phi^{-1} \zeta_{P'}(S), & y \neq 0, \\ S, & y = 0, \end{cases}$$

where S denotes the segment $-1 \leq t \leq 1$. The path $C(P)$ depends continuously on P (also at $y = 0$) and has the property that $x + iyt \in \eta$ for $t \in C(P)$ and $P \in D(\eta)$. Every point $P_0 \in D(\eta)$ possesses a neighborhood N such that $\zeta_P(C(P_0)) \in \eta$ for $P \in N$. We may therefore replace $C(P)$ by $C(P_0)$ for $P \in N$ and conclude by a classical theorem that (2) is an analytic function of P at $P = P_0$. The remaining assertions of Theorem I can be verified as in [2].

For $\operatorname{Re} \nu \leq 0$ the path $C(P)$ has to be replaced by a figure eight loop as used by Mackie [4]. The idea of the proof remains the same.

Remarks. 1. Theorem I sets up a mapping $u = \theta[g]$ of the class of all functions $g(z)$ regular in η into the class of all solutions $u(x, y)$ of (1) regular in $D(\eta)$. For $\operatorname{Re} \nu > 0$ and for special domains η we can show that this mapping is onto (and thus 1-1); indeed, it is onto a seemingly less restricted class of solutions. This is implied by

THEOREM II. *Let $\operatorname{Re} \nu > 0$, and let the solution $u(x, y)$ be real-analytic in a simply connected domain η which is symmetric with respect to $y = 0$. Then the function $g(x) = u(x, 0)$ can be continued analytically to a function $g(z)$ which is holomorphic for $z \in \eta$.*

For convex domains this theorem is a special case of Satz I, in [2]. The proof given there is also valid for nonconvex domains.

Szegö [5] has recently proved a theorem which in slightly extended

form can be stated as follows: Let $g(z)$ be regular in $|z| < r$, and let $u(x, y) = \theta[g]$, where $\nu = 1/2$. Then, for any real ϑ , $(x, y) = (r \cos \vartheta, r \sin \vartheta)$ is a regular point of $u(x, y)$ if and only if $z = re^{i\vartheta}$ and $\bar{z} = re^{-i\vartheta}$ are regular points of $g(z)$. This is clearly a corollary of the logical product of the Theorems I and II for convex regions η . This case, however, was already covered in our earlier paper [2].

2. If the function ϕ used in the proof of Theorem I is sufficiently smooth, the length of $C(P)$ depends continuously on P . It follows that the mapping θ is continuous in the following sense: -

THEOREM III. For every $\epsilon > 0$ and every closed subdomain η of $\bar{\eta}$ there exists a $\delta > 0$ such that $|\theta[g(z)]| < \epsilon$ in $D(\bar{\eta})$ if $|g(z)| < \delta$ for $z \in \bar{\eta}$.

It follows by Runge's theorem that every $u(x, y)$ satisfying the hypothesis of Theorem II can be approximated uniformly in every $D(\bar{\eta})$ by linear combinations of the functions $\theta[z^n]$, $n = 0, 1, 2, \dots$. These functions can be expressed explicitly in terms of ultraspherical polynomials (see [2]).

3. We finally mention that the considerations of this note can be extended with little change to solutions of the equation obtained from (1) by adding a term $+k^2u$ (k a complex constant) to the left. The role of (2) is then taken over by

$$u(x, y) = \frac{\Gamma(\nu + 1/2)}{\Gamma(1/2)} \int_{-1}^1 g(x + iyt) (ky/2)^{-\nu+1} J_{\nu-1}(ky(1-t^2)^{1/2}) dt,$$

where $J_{\nu-1}$ is the Bessel function of order $\nu - 1$.

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AMERICAN UNIVERSITY⁶

⁶ The author is now at the University of California, Los Angeles.