

A CONDITION THAT A NORMAL OPERATOR HAVE A CLOSED NUMERICAL RANGE

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In this paper the following theorem will be proved: *A necessary and sufficient condition that the numerical range of a bounded normal operator T in a Hilbert space is closed is that the set of extreme points of the convex closure of the spectrum of T contains no points of the continuous spectrum of T .*

A few terms should be properly defined. The numerical range of a linear bounded operator T in a Hilbert space is defined as the set of all complex numbers of the form (Tx, x) where x is a unit vector in the space. Clearly the numerical range of T contains the point spectrum of T . Stone [2] proved that the numerical range of a bounded linear operator T is convex, and that if T is normal then the closure of its numerical range is exactly the convex closure $K(T)$ of the spectrum of T . A point λ of $K(T)$ is defined as an extreme point of $K(T)$ if no line segment joining any two points of $K(T)$ other than λ contains λ . Denote by A the set of extreme points of $K(T)$. It is easy to see that A is contained in the spectrum of T , and that the convex closure of A is exactly $K(T)$. Hence the sufficiency of the condition follows readily from these facts.

The necessity of the condition can be proved by contradiction.

Suppose that there exists a point a_0 in A such that a_0 lies in the continuous spectrum of T . Now the numerical range of T is closed, and hence a_0 lies in it. So we can find a unit vector x in the space such that $a_0 = (Tx, x)$. Since T is bounded and normal, it has a resolution of the identity $\{E_\lambda\}$, and T can be represented as an integral with respect to $\{E_\lambda\}$, [1], i.e.,

$$T = \int_G \lambda dE_\lambda$$

where G is any large circle containing the spectrum of T in its interior. Therefore we have:

$$a_0 = (Tx, x) = \int_G \lambda d\|F_\lambda x\|^2.$$

For each $n = 1, 2, 3, \dots$, consider the open circle D_n about the

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point a_0 with radius $1/2^n$, and denote by $X_n(\lambda)$ the characteristic function of D_n , i.e., $X_n(\lambda) = 1$ for λ in D_n , and $= 0$ otherwise. Since every Borel set is measurable with respect to $\{E_\lambda\}$, [1], for each $n = 1, 2, 3, \dots$, we have

$$\begin{aligned} I &= \int_G dE_\lambda \\ &= \int_G X_n(\lambda) dE_\lambda + \int_G [1 - X_n(\lambda)] dE_\lambda \\ &= \int_{G_n} X_n(\lambda) dE_\lambda + \int_{R_n} [1 - X_n(\lambda)] dE_\lambda, \end{aligned}$$

where G_n is any convex region containing D_n in its interior, and R_n is any region containing in its interior the set of the spectrum of T less D_n ; in particular, one can choose R_n such that a_0 is at a positive distance outside of R_n and R_n is convex. This can be done because a_0 is an extreme point of $K(T)$. Let for each $n = 1, 2, 3, \dots$,

$$F_{1,n} = \int_{G_n} X_n(\lambda) dE_\lambda, \quad F_{2,n} = \int_{R_n} [1 - X_n(\lambda)] dE_\lambda,$$

then $F_{1,n}$ and $F_{2,n}$ are projection operators and are orthogonal to each other. It is claimed that for each $n = 1, 2, 3, \dots$, $F_{2,n}x = \theta$. Suppose not, then let $c_{i,n} = \|F_{i,n}x\|^2$ for $i = 1, 2$. So we have for some n , $c_{2,n} > 0$, and

$$1 = \|F_{1,n}x + F_{2,n}x\|^2 = \|F_{1,n}x\|^2 + \|F_{2,n}x\|^2 = c_{1,n} + c_{2,n}.$$

Hence $0 \leq c_{1,n} < 1$ and $0 < c_{2,n} \leq 1$. Since the regions G_n and R_n are convex, we may apply the mean value theorem and deduce that

$$\begin{aligned} a_0 &= (Tx, x) = \int_G \lambda d\|E_\lambda x\|^2 \\ &= \int_{G_n} \lambda X_n(\lambda) d\|E_\lambda x\|^2 + \int_{R_n} \lambda [1 - X_n(\lambda)] d\|E_\lambda x\|^2 \\ &= \lambda_{1,n} c_{1,n} + \lambda_{2,n} c_{2,n} \end{aligned}$$

where $\lambda_{1,n}$ lies in G_n , and $\lambda_{2,n}$ lies in R_n . Since R_n is so chosen that it is at a positive distance away from a_0 , therefore $\lambda_{2,n}$ is not equal to a_0 which implies that $0 < c_{1,n} < 1$, $0 < c_{2,n} < 1$, and $\lambda_{1,n} \neq a_0$. Hence both $F_{1,n}x$ and $F_{2,n}x$ are not zero vectors. Consider the unit vector $y_{2,n} = F_{2,n}x / \|F_{2,n}x\|$; then we have

$$\begin{aligned}
(Ty_{2,n}, y_{2,n}) &= \int_G \lambda d\|E_\lambda y_{2,n}\|^2 \\
&= \frac{1}{\|F_{2,n}x\|^2} \int_G \lambda d\|E_\lambda F_{2,n}x\|^2 \\
&= \frac{1}{c_{2,n}} \int_G \lambda d \int_{R_n} [1 - X_n(\mu)] d\|E_\lambda E_\mu x\|^2 \\
&= \frac{1}{c_{2,n}} \int_{R_n} \lambda [1 - X_n(\lambda)] d\|E_\lambda x\|^2 = \lambda_{2,n}.
\end{aligned}$$

Hence $\lambda_{2,n}$ lies in the numerical range of T . Similarly, $\lambda_{1,n}$ lies in the numerical range of T . Now by hypothesis the numerical range is closed, and T is normal, and consequently the numerical range is exactly $K(T)$. However, we have

$$a_0 = c_{1,n}\lambda_{1,n} + c_{2,n}\lambda_{2,n}$$

with $c_{1,n} + c_{2,n} = 1$, $0 < c_{1,n}, c_{2,n} < 1$, and $\lambda_{1,n} \neq a_0 \neq \lambda_{2,n}$. Furthermore a_0 is an extreme point of $K(T)$. So this is a contradiction.

Therefore we conclude that $F_{2,n}x$ is a zero vector for all $n = 1, 2, 3, \dots$, which implies that $F_{1,n}x = x$ for all $n = 1, 2, 3, \dots$. Now for all $n = 1, 2, 3, \dots$,

$$\begin{aligned}
Tx &= \int_G \lambda dE_\lambda x = \int_G \lambda dE_\lambda F_{1,n}x \\
&= \int_G \lambda d \int_{G_n} X_n(\mu) dE_\lambda E_\mu x \\
&= \int_{G_n} \lambda X_n(\lambda) dE_\lambda x,
\end{aligned}$$

and

$$\begin{aligned}
\|Tx - a_0x\| &= \left\| \int_{G_n} \lambda X_n(\lambda) dE_\lambda x - a_0 \int_{G_n} X_n(\lambda) dE_\lambda x \right\| \\
&= \left\| \int_{G_n} (\lambda - a_0) X_n(\lambda) dE_\lambda x \right\| \\
&\leq \max_{\lambda \in D_n} |\lambda - a_0| \leq \frac{1}{2^n} \rightarrow 0
\end{aligned}$$

as $n \rightarrow \infty$. Therefore $Tx = a_0x$ which implies that a_0 lies in the point spectrum of T . However the point spectrum and the continuous spec-

trum of T are mutually exclusive, and consequently we have a contradiction. Hence the theorem is proved. The following corollary is an immediate consequence of the theorem:

The numerical range of a unitary operator in a Hilbert space is closed if and only if the spectrum of the unitary operator consists entirely of the point spectrum.

REFERENCES

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ON POSTULATES FOR GENERAL QUANTUM MECHANICS

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A collection of postulates for the observable quantities in a general quantum mechanical system has been given by Segal, [4]. In this system, the observables form a real Banach space S which is partially ordered by a closed convex positive cone P , and which has a unit element e interior to the positive cone, all satisfying $P \cap (e - P) = S_1(e) \cap S_1(0)$, where $S_r(x)$ is the solid sphere of radius r , center x . Alternatively, S is a partially ordered linear space with an element $e > 0$ which is a Banach space under the norm $\|x\| = \min \{\lambda: -\lambda e \leq x \leq \lambda e\}$, where the minimum is attained. Let us call such a space a *partially ordered Banach space with unit*. We shall show that any such Banach space S with order unit e can be given the algebraic structure postulated by Segal, in possibly different ways. In this proof it is very useful to have an alternative formulation of Segal's postulates stated in terms of the partial ordering alone. We are then able to give examples of such systems with various pathologies and to classify the three dimensional ones. Finally we shall give a necessary and sufficient condition that a system of operators satisfying these postulates be a C^* special Jordan algebra: the system of hermitean elements of a C^* algebra.

DEFINITION. *A spectral function L on a partially ordered Banach space S with order unit e is a function assigning to each point $x \in S$ a*

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