ON n -METAHARMONIC FUNCTIONS AND HARMONIC FUNCTIONS OF INFINITE ORDER

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Introduction. Let E^N denote the space of N real variables $x = (x_1, \dots, x_N)$ with norm $|x| = (\sum x_i^2)^{1/2}$. A function $u(x)$ defined in a domain $D \subset E^N$, is called n -metaharmonic in D , if it satisfies $p(\Delta)u(x) = 0$ in D , where $p(t)$ is a polynomial in t of degree n and $\Delta = \sum \partial^2 / \partial x_i^2$. In this paper we prove a representation theorem for n -metaharmonic functions. The special case of $p(t) = t^n$ was already proved by Tolotti [5]. As an application of the representation theorem we prove in §2 a generalization of Liouville's Theorem for n -metaharmonic functions. In §3 we prove a generalization of Liouville's Theorem for harmonic functions of infinite order. We recall the definition first given by Aronszajn in [1; 2]:

A function $u(x)$ defined and infinitely differentiable in a domain D is said to be harmonic of infinite order in D , if

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} (|\Delta^n u(x)|)^{1/n} = 0$$

uniformly in some neighbourhood of every point $x \in D$.

1. Definition. A domain $D \subset E^N$ is said to possess property A, if there exists a hyperplane $x_1 = x_1^0$ such that any straight line parallel to the x_1 -axis which intersects D , intersects it in an interval having a point on the hyperplane.

THEOREM 1. *Let $p(t)$ be a polynomial with real coefficients having the following decomposition:*

$$(1) \quad p(t) = \prod_{i=1}^k (t - \alpha_i)^{n_i} \prod_{i=1}^l (t - \beta_i)^{m_i} (t - \bar{\beta}_i)^{m_i},$$

where α_i, β_i are real and complex respectively. Consider the linear elliptic equation

$$(2) \quad p(\Delta)u(x) = 0$$

in a domain D possessing property A. Then the general real solution of (2) in any domain $D' \supset \bar{D}$, where $\bar{D}$ is the closure of D , has the following representation in D :

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$$(3) \quad u(x) = \sum_{i=1}^k \sum_{j=0}^{n_i-1} x_1^j u_j^{\alpha_i}(x) + \sum_{i=1}^l \sum_{j=0}^{m_i-1} x_1^j \operatorname{Re}\{u_j^{\beta_i}(x) + \bar{u}_j^{\bar{\beta}_i}(x)\},$$

where $u_j^\gamma(x)$ denote solutions of $(\Delta - \gamma)u(x) = 0$ in D .

More precisely, we mean:

- (i) the u given by (3) is a solution of (2), and
- (ii) for every solution u of (2) in D' , there exist functions $u_j^{\alpha_i}, u_j^{\beta_i}, \bar{u}_j^{\bar{\beta}_i}$ such that (3) holds in D .

1.1. We give the proof for $N=2$, that is when $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, the proof of the general case is analogous. We first prove the theorem in the case of

$$(4) \quad p(t) = (t - \alpha)^n, \quad \alpha \text{ real.}$$

The representation (3) becomes

$$(5) \quad u = u_0 + x u_1 + \cdots + x^{n-1} u_{n-1},$$

where $(\Delta - \alpha)u_i = 0$.

Now it can be proved by induction that if a function $v(x)$ satisfies $(\Delta - \alpha)v = 0$, then

$$(6) \quad (\Delta - \alpha)^m (x^m v) = m! \frac{\partial^m v}{\partial x^m}.$$

It follows that the function u given by (5) is a solution of $(\Delta - \alpha)^n u = 0$. Conversely let $u(x)$ be a solution of $(\Delta - \alpha)^n u = 0$ in D' . If we prove the existence of a function $v(x)$ in a domain $D'' \supset \bar{D}$, such that

$$(7) \quad (\Delta - \alpha)v = 0, \quad (\Delta - \alpha)^{n-1}(x^{n-1}v) = (\Delta - \alpha)^{n-1}u,$$

then the existence of the desired representation followed by induction. Take D'' to be a domain possessing property A and such that $D'' \supset \bar{D}$, $\bar{D}'' \subset D'$ and define in D''

$$(8) \quad v_0(x, y) = \frac{1}{(n-1)!} \int_{x_1^0}^x \int_{x_1^0}^{x_{n-2}} \cdots \cdot \int_{x_1^0}^{x_1} (\Delta - \alpha)^{n-1} u(x_0, y) dx_0 dx_1 \cdots dx_{n-2}.$$

Since

$$\frac{\partial^{n-1}}{\partial x^{n-1}} (\Delta - \alpha)v_0(x, y) = \frac{1}{(n-1)!} (\Delta - \alpha)^n u(x, y) \equiv 0,$$

we have

$q(t)$ is the sum of factors $g_i(t)$ of $p(t)$. Denoting

$$h_i(t) = \frac{\dot{p}(t)}{g_i(t)}, \quad u_i = A^{-1}g_i(\Delta)u,$$

we have $h_i(\Delta)u_i = 0$. Since $u = \sum u_i$, it is sufficient to show that each u_i is the sum of terms which appear in (10).

If $t - \alpha_k$ is a factor of $g_i(t)$, this follows from the inductive assumption on $n_k - 1$. Otherwise, the $(k-1)$ th equality in (12) contributes to $g_i(t)$ the factor $(t - \alpha_{k-1})^{n_{k-1}-1}$, in which case:

If the $(k-2)$ th equality in (12) contributes to $g_i(t)$ the factor $t - \alpha_{k-1}$, we can apply the inductive assumption on $k-1$. Otherwise $(t - \alpha_{k-2})^{n_{k-2}-1}$ is a factor of $g_i(t)$.

Continuing in this way, we finally reach the case where $(t - \alpha_1)^{n_1}$ is a factor of $g_i(t)$ and we can apply the inductive assumption on $k-1$.

1.3. From the arguments in 1.1, 1.2 it is obvious that the general complex solution of (2) has the form (3), with complex $u_j^{\alpha_i}$, $u_j^{\beta_i}$, $u_j^{\gamma_i}$. (This is true also in the case that the coefficients of $p(t)$ are complex numbers.) Taking real parts, we arrive at the desired representation.

2. THEOREM 2. *Let $p(t)$ be a polynomial with real coefficients, having no negative roots and let $u(x)$ be a solution in E^N of*

$$(1) \quad p(\Delta)u(x) = 0.$$

If $u(x)$ is bounded in E^N , it is necessarily a constant.

2.1. We give the proof in the case $N=3$ and $p(t)$ having only real roots, that is

$$(2) \quad p(t) = t^k \prod_{i=1}^l (t - \beta_i)^{l_i}, \quad \beta_i > 0$$

The proof of the general case is analogous.

Let x^0 be an arbitrary point which we may assume to be the origin. We shall apply the formula [4, p. 259]

$$(3) \quad \frac{1}{4\pi R^2} \int_{S_R} u(x) dS = \sum_{n=0}^m \frac{R^{2n} \Delta^n u(x^0)}{(2n+1)!} + \frac{1}{2\pi(2m+1)!} \int_{K_R} \frac{(R-r)\Delta^{m+1}u(x)}{Rr} dV,$$

where K_R is a sphere of radius R , centre x^0 and surface S_R , and $r = |x - x^0|$. Using Theorem 1, we have a representation of $u(x)$ in K_R ,

$$(4) \quad u(x) = \sum_{j=0}^{k-1} x_1^j u_j^0(x) + \sum_{i=1}^l \sum_{j=0}^{l_i-1} x_1^j u_j^{\beta_i}(x).$$

Observe that

$$(5) \quad \Delta^n(x_1 v) = n \frac{\partial}{\partial x_1} \Delta^{n-1} v + x_1 \Delta^n v$$

and that

$$\Delta^n u_j^{\beta_i} = \beta_i^n u_j^{\beta_i},$$

so that

$$|\Delta^n x_1^j u_j^{\beta_i}(x)| \leq H \beta_i^n n^j, \quad n = 1, 2, \dots$$

for $x \in K_R$, where H is some constant. It follows that the volume integral in (3) approaches zero as $m \rightarrow \infty$.

Using (5) we obtain a formula for $\Delta^n x_1^j u_j^{\beta_i}$ at the point x^0 in terms of $u_j^{\beta_i}$ and its derivatives up to the order $j-1$ at the point x^0 . Letting $m \rightarrow \infty$ in (3) and denoting

$$(6) \quad A_j(t) = \sum_{n=j}^{\infty} \frac{n(n-1) \cdots (n-j+1)}{(2n+1)!} t^{2n},$$

we obtain after some elementary calculations

$$(7) \quad \frac{1}{4\pi R^2} \int_{S_R} u(x) dS = \sum_{j=0}^{k-1} a_j R^{2j} + \sum_{i=1}^l \sum_{j=0}^{l_i-1} b_j^{\beta_i} A_j((\beta_i)^{1/2} R),$$

where $a_j, b_j^{\beta_i}$ are some coefficients.

Equation (7) clearly holds with every positive $R' < R$ instead of R . It is easily verified that

$$(8) \quad A_j((t)^{1/2}) = t^j \frac{d^j}{dt^j} \left(\frac{\sinh t^{1/2}}{t^{1/2}} \right),$$

so that the functions $R^{2j}, A_j((\beta_i)^{1/2} R)$ are linearly independent.

It follows that (7) is true for every positive R . Taking $R \rightarrow \infty$ and comparing the behavior at infinity of the functions which appear in (7), we conclude that $a_j = 0$ $1 \leq j \leq k-1$; $b_j^{\beta_i} = 0$ $1 \leq i \leq l$ $0 \leq j \leq l_i-1$. Equation (7) reduces to

$$(9) \quad \frac{1}{4\pi R^2} \int_{S_R} u(x) dS = a_0.$$

Taking $R \rightarrow 0$ in (9), we see that $a_0 = u(x^0)$. Since x^0 is arbitrary, $u(x)$ is harmonic and therefore it is constant.

2.2. From the proof of Theorem 2, the following consequence can easily be deduced:

THEOREM 3. *Let $p(t)$ be a polynomial of degree n , with real coefficients, having no negative roots and $p(0) \neq 0$. Denote by α_i , $i=1, \dots, n$ the roots of $p(t)$ and let $B = \min |\operatorname{Im} (-\alpha_i)^{1/2}|$. If $u(x)$ is a solution in E^N of $p(\Delta)u(x) = 0$ and if*

$$(10) \quad u(x) = o(|x|^{(1-N)/2} e^{B|x|}) \quad \text{as } |x| \rightarrow \infty,$$

then $u(x)$ is identically zero.

Note that condition (10) is sharp.

2.3. The method used in the proof of Theorem 2, can be used to prove Liouville's Theorem for general $p(t)$.

THEOREM 4. *Let $p(t)$ be a polynomial with real coefficients and let $u(x)$ be a solution in E^N of $p(\Delta)u(x) = 0$. Suppose further*

- (a) $\lim_{|x| \rightarrow \infty} u(x)$ exists, and
- (b) for every $x^0 \in E$,

$$(11) \quad \lim_{R \rightarrow \infty} R^{(N-1)/2} \left| \frac{1}{|S|} \int_{S_R} u(x) dS - \lim_{|z| \rightarrow \infty} u(x) \right| \text{ exists,}$$

where S_R is the surface of a sphere of radius R and centre x^0 , and $|S|$ is the area of S_R .

Then $u(x)$ is necessarily a constant.

Note that conditions (a) (b) are sharp. It is not sufficient to assume merely the boundedness of

$$R^{(N-1)/2} \left| \frac{1}{|S|} \int_{S_R} u(x) dS - \lim_{|z| \rightarrow \infty} u(x) \right| \quad (R \rightarrow \infty),$$

as is shown by the example of

$$N = 3, \quad p(t) = t + 1, \quad u(x) = \sin |x| / |x|.$$

3. **THEOREM 5.** *Let $u(x)$ be a harmonic function of infinite order in E^N , satisfying for every $x \in E^N$*

$$(1) \quad \lim_{n \rightarrow \infty} (|\Delta^n u(x)|)^{1/n} = 0.$$

Then if $u(x)$ is bounded in E^N , it is necessarily a constant.

Note that condition (1) is sharp. If we assumed

$$\limsup_{n \rightarrow \infty} (|\Delta^n u(x)|)^{1/n} \leq \epsilon,$$

for some positive ϵ , the assertion would not hold, as is shown by the counterexample of

$$u(x) = \sin \epsilon^{1/2} x_1.$$

3.1. We give the proof in the case of $N=3$. The proof of the general case is analogous.

Taking in 2.1 (3) $m \rightarrow \infty$, we obtain

$$(2) \quad \frac{1}{4\pi R^2} \int_{S_R} u(x) dS = \sum_{n=0}^{\infty} \frac{R^{2n}}{(2n+1)!} \Delta^n u(x^0).$$

Denote

$$(3) \quad a_{2n} = \Delta^n u(x^0)/(2n+1)!,$$

then the function

$$(4) \quad g(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n}$$

is an integral function. It is bounded on the real axis, as follows from (2) and from the boundedness of $u(x)$.

If we show that $g(z)$ is either of order <1 , or of minimal type of the order 1, then applying a known theorem of Phragmén (see [6, p. 178]), we conclude that $g(z) \equiv \text{const.}$, which implies $3!a_2 = \Delta u(x^0) = 0$. Since x^0 is arbitrary, $u(x)$ is harmonic and therefore $u(x) \equiv \text{const.}$

Now according to a well known criterion concerning the type and order of an integral function (see [3, p. 238]), all we need to prove is that

$$(5) \quad \lim_{n \rightarrow \infty} n(|a_{2n}|)^{1/2n} = 0.$$

But (5) follows immediately from (3), (1).

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