A GENERALIZATION OF HELLY'S THEOREM

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1. The purpose of this note is to present a generalization, stated as Theorem 2, of Helly's theorem on the intersection of convex sets. Throughout this paper, E^n will denote a euclidean space of n dimensions, and for any set $A \subset E^n$, $H(A)$ will denote the convex hull of A ; i.e., the smallest convex set containing A . Furthermore, by $d(A) = r$ we will mean that $H(A)$ contains a linear manifold of dimension r , but no higher dimension. Finally, for any linear manifold P , P^* will denote the homogeneous linear manifold of dim $d(P)$ parallel to P .

It is well known, [1; 2], that Helly's theorem is essentially a consequence of the following fact:

Any set of $n+2$ points in E^n can be partitioned into two nonempty disjoint sets A and B in such a way that $H(A) \cap H(B) \neq \emptyset$.

In proving Theorem 2, we shall first establish Theorem 1, which is a generalization of the above proposition.

2. THEOREM 1. *Let n, k be two integers such that $2 \leq k \leq n+2$. In E^n , let $P_1, \dots, P_k$ be k linear manifolds, each of dimension $n-k+2$. Then the P_i can be arranged so that*

$$H\left(\bigcup_{i=1}^t P_i\right) \cap H\left(\bigcup_{i=t+1}^k P_i\right)$$

contains a linear manifold of dimension $n-k+2$, where t is an integer satisfying $0 < t < k$.

Before proving Theorem 1, we will state without proof two lemmas. The first is a special case of the proposition of Paragraph 1.

LEMMA 1. *In E^r , let $p_1, \dots, p_{r+1}$ be the vertices of a barycentric coordinate system, and let*

$$p_0 = \sum_{i=1}^{r+1} \gamma_i p_i, \quad \sum_{i=1}^{r+1} \gamma_i = 1, \quad \gamma_i \neq 0.$$

Then there exist α_i ($0 \leq i \leq r+1$) satisfying

$$\sum_{i \in C} \alpha_i p_i = \sum_{i \in D} \alpha_i p_i, \quad \sum_{i \in C} \alpha_i = \sum_{i \in D} \alpha_i = 1, \quad \alpha_i > 0$$

$0 \in C$, $C \neq \emptyset \neq D$, $C \cup D = \{0, \dots, r+1\}$, $C \cap D = \emptyset$ and furthermore,

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$$\gamma_i > 0 \rightarrow i \in D.$$

LEMMA 2. In E^r , let $p_1, \dots, p_{r+1}$ be the vertices of a barycentric coordinate system, and suppose $p_k = \sum_{i=1}^{r+1} \beta_i p_i$, $\sum_{i=1}^{r+1} \beta_i = 1$. Then for each index i such that $\beta_i \neq 0$, the points $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{r+1}, p_k$ are affine linearly independent.

PROOF OF THEOREM 1. The theorem is obviously true for $k=2$ or $k=n+2$. Hence, we will assume it is true for $n-1, k-1$, and prove it true for n, k . We may further assume that no subspace $E^{n-1} \subset E^n$ contains all except possibly one of the P_i , for then we would have, in E^{n-1} , $k-1$ linear manifolds each of dimension $(n-1)-(k-1)+2 = n-k+2$ and the theorem would be true by our induction hypothesis.

Now let $d(\cup P_i) = n-r$ ($1 \leq i \leq k$, $0 \leq r \leq k-2$). Let L be a linear manifold of dimension $n-r$ contained in $H(\cup P_i)$, and M an r -dimensional linear manifold orthogonal to L . Finally, let p_i be the projection of P_i on M .

Clearly, there must be $r+1$ affine linearly independent points among the p_i 's, say $p_1, \dots, p_{r+1}$. Let us now write $A = \{1, \dots, r+1\}$, $B = \{r+2, \dots, k\}$. We may assume A to be so chosen that

$$d\left(\bigcup_{i \in B} P_i\right) = n-s \text{ is maximal,} \quad r \leq s \leq k-2.$$

Note that after selecting any one P_i ($i \in B$), we need at most

$$(n-s) - (n-k+2) = k-s-2$$

more of the P_i 's to provide the $k-s-2$ additional dimensions. Hence, if $r < s$, we have

$$[k - (r+1)] - [1 + (k-s-2)] = s-r > 0$$

and we may assume that

$$P_k + q_k \subset H\left(\bigcup_{k \neq i, i \in B} P_i\right)$$

where q_k is a fixed point.

In any case, using the p_i ($i \in A$) as vertices, we may write $p_k = \sum_{i \in A} \beta_i p_i$, $\sum_{i \in A} \beta_i = 1$, and we may further assume $\beta_{r+1} > 0$. Now let P_0 be a linear manifold of dimension $n-s$ contained in the interior of $H(\cup_{i \in B} P_i)$, such that the projection of P_0 on M is

$$p_0 = \sum_{i \in A} \gamma_i p_i, \quad \sum_{i \in A} \gamma_i = 1.$$

We may assume $\gamma_i \neq 0$ ($i \in A$), since $\gamma_{i_0} = 0$ for every choice of P_0 would imply that

$$H(\cup P_i) \subset E^{n-1} \quad \text{for } i = 1, \dots, i_0 - 1, i_0 + 1, \dots, k$$

and this is impossible. In particular, we may assume $\gamma_{r+1} > 0$.

Now using Lemma 1, we have

$$\sum_{i \in C} \alpha_i p_i = \sum_{i \in D} \alpha_i p_i, \quad \sum_{i \in C} \alpha_i = \sum_{i \in D} \alpha_i = 1, \quad \alpha_i > 0,$$

$$r+1 \in D, 0 \in C, C \neq \emptyset \neq D, C \cup D = \{0, \dots, r+1\}, C \cap D = \emptyset.$$

We assert that $P_{r+1}^* \subset P_0^*$. This is clearly true if $r=s$. If $r \neq s$, then from $\beta_{r+1} > 0$ and Lemma 2, we could have interchanged P_{r+1} and P_k . Hence, from the maximality of $n-s$ we must have $P_{r+1}^* \subset P_0^*$.

Now choose a vector basis $v_1, \dots, v_n$ in E^n such that $v_1, \dots, v_{n-k+2}$ span P_{r+1}^* , and $v_1, \dots, v_{n-s}$ span P_0^* . Then clearly

$$v_j + q_i \subset H(\cup P_i) \quad 1 \leq i \leq r, n-s+1 \leq j \leq n-r$$

where the q_j are fixed points. Then we may write

$$P_0 = \{(x_1, \dots, x_{n-s}, a_{0,n-s+1}, \dots, a_{0,n})\},$$

$$P_{r+1} = \{(y_1, \dots, y_{n-k+2}, a_{r+1,n-k+3}, \dots, a_{r+1,n})\}$$

where the x_i and y_i vary over all real values, and the $a_{0,j}$ and $a_{r+1,j}$ are constants.

Now select any fixed point $(a_{ij}) \in P_i$, $0 \leq i \leq r+1$, and define

$$Q = \{(z_1, \dots, z_{n-k+2}, b_{n-k+3}, \dots, b_n)\}$$

where the z_i vary over all real values, and

$$\text{for } n-k+3 \leq j \leq n-s, \quad b_j = \sum_{i \in D} \alpha_i a_{ij},$$

$$\text{for } n-s+1 \leq j \leq n-r, \quad b_j = \sum_{i \in D} \alpha_i a_{ij}, \quad \text{if } v_j + q_i \not\subset H\left(\cup_{i \in D} P_i\right),$$

$$b_j = \sum_{i \in C} \alpha_i a_{ij}, \quad \text{otherwise,}$$

$$\text{for } n-r+1 \leq j \leq n, \quad b_j = \sum_{i \in C} \alpha_i a_{ij}.$$

Let (b_j) be any fixed point in Q . We have already shown that

$$b_j = \sum_{i \in C} \alpha_i a_{ij} = \sum_{i \in D} \alpha_i a_{ij} \quad n-r+1 \leq j \leq n.$$

Then, since the $\alpha_i \neq 0$, we can always choose the x_j ($1 \leq j \leq n-s$), y_j ($1 \leq j \leq n-k+2$) and some a_{ij} ($n-s+1 \leq j \leq n-r$) so that

$$b_j = \sum_{i \in C} \alpha_i a_{ij} = \sum_{i \in D} \alpha_i a_{ij} \quad 1 \leq j \leq n.$$

Hence we have

$$Q \subset H\left(\bigcup_{0 < i \in B \cup C} P_i\right) \cap H\left(\bigcup_{i \in D} P_i\right).$$

This concludes the proof of Theorem 1.

3. We now come to our main result, which includes Helly's theorem as the special case corresponding to $k = n + 1$.

THEOREM 2. *In E^n , let $Q_1, \dots, Q_r$ be r convex sets. Let k be an integer satisfying $1 \leq k \leq n + 1$, $k \leq r$. If the intersection of any k of the Q_i 's contains a linear manifold of dimension $n - k + 1$, then $\bigcap_{i=1}^r Q_i$ contains a linear manifold of dimension $n - k + 1$.*

PROOF. The theorem is trivial for $r = k$. To prove the theorem by induction on r , we may assume that $k < r$, and that any $r - 1$ of the Q_i have an intersection which contains a linear manifold of dimension $n - k + 1$. Denote these linear manifolds by P_j , where

$$P_i \subset Q_i \quad \text{for } i \neq j \quad 1 \leq i, j \leq r.$$

If $P_i = P_j$ ($i \neq j$), then the theorem is proved. If, however, the P_j are distinct, then by Theorem 1 we may assume that for some integer t satisfying $0 < t \leq k$, the set

$$H\left(\bigcup_{j=1}^t P_j\right) \cap H\left(\bigcup_{j=t+1}^{k+1} P_j\right)$$

contains a linear manifold W of dimension $n - k + 1$. Hence for $i > t$ we have

$$P_i \subset Q_i \quad 1 \leq j \leq$$

and since Q_i is convex

$$H\left(\bigcup_{j=1}^t P_j\right) \subset Q_i.$$

Similarly for $i \leq t$ we have

$$H\left(\bigcup_{j=t+1}^{k+1} P_j\right) \subset Q_i.$$

Therefore

$$W \subset H\left(\bigcup_{j=1}^t P_j\right) \cap H\left(\bigcup_{j=t+1}^{k+1} P_j\right) \subset \bigcap_{i=1}^r Q_i$$

and the proof is complete.

It is easy to see that our final result may also be formulated as

THEOREM 2'. Let E be a real vector space (finite or infinite dimensional) and let $P_1, \dots, P_N$ ($k \leq N < \infty$) be convex sets in E such that the intersection of any k of the P_i contains a subspace of codimension $< k$. Then $\bigcap_{i=1}^N P_i$ contains a subspace of codimension $< k$.

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