ON THE INDEX OF A FIBERED MANIFOLD¹

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Introduction. Let V be a real vector space of dimension r . Let $F(x, y) = \langle x, y \rangle$, $x, y \in V$, be a real-valued symmetric bilinear function. We can find a base e_i , $1 \leq i \leq r$, in V , such that

$$(1) \quad F(x, y) = \sum_{i=1}^p x^i y^i - \sum_{i=p+1}^{p+q} x^i y^i$$

where $x = \sum_{i=1}^r x^i e_i$ and $y = \sum_{i=1}^r y^i e_i$.

The number $p - q$ is called the index of F , to be denoted by $\tau(F)$. It depends only on F . If F is nonsingular (i.e. $p + q = r$), then $\min(p, q)$ equals the maximal dimension of the linear subspaces of V contained in the "cone" $F(x, x) = 0$.

Now let M be a compact oriented manifold. The index of M is defined to be zero, if the dimension of M is not a multiple of 4. If M has the dimension $4k$, consider the cohomology group $H^{2k}(M)$ with real coefficients. This is a real vector space, and the equation

$$(2) \quad \langle x, y \rangle \xi = x \cup y, \quad x, y \in H^{2k}(M),$$

where ξ is the generator of $H^{4k}(M)$ defined by the given orientation of M , defines a real-valued symmetric bilinear form $\langle x, y \rangle$ over $H^{2k}(M)$. Its index is called the index of M , to be denoted by $\tau(M)$. Reversal of the orientation of M changes the sign of the index. The form $\langle x, y \rangle$ defined by (2) is nonsingular, since, by Poincaré's duality theorem, the equation $x \cup y = 0$ for all $x \in H^{2k}(M)$ implies $y = 0$.

The main purpose of this paper is to prove the theorem:

THEOREM. *Let $E \rightarrow B$ be a fiber bundle, with the typical fiber F , such that the following conditions are satisfied:*

- (1) *E, B, F are compact connected oriented manifolds;*
- (2) *The fundamental group $\pi_1(B)$ acts trivially on the cohomology ring $H^*(F)$ of F .*

Then, if E, B, F are oriented coherently, so that the orientation of E is induced by those of F and B , the index of E is the product of the indices of F and B , that is,

$$\tau(E) = \tau(F)\tau(B).$$

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REMARK. We do not know whether condition (2) and the connectedness hypothesis of condition (1) are necessary. For instance, let E be an n -sheeted covering of B (the spaces B and E still being compact oriented manifolds); is it true that $\tau(E) = n\tau(B)$? We know the answer to be positive only when B possesses a differentiable structure: in that case, according to a theorem of one of us, $\tau(B)$ (resp. $\tau(E)$) is equal to the Pontrjagin number $L(B)$ (resp. $L(E)$) and it is clear that $L(E) = n \cdot L(B)$.

1. **Algebraic properties of the index of a matrix.** Let e_i , $1 \leq i \leq r$, be a base in V . A real-valued symmetric bilinear function $\langle x, y \rangle$ defines a real-valued symmetric matrix $C = (c_{ij})$, $c_{ij} = \langle e_i, e_j \rangle$, $1 \leq i, j \leq r$, and is determined by it. The index of the bilinear function is equal to the index $\tau(C)$ of C , if we define the latter to be the excess of the number of positive eigenvalues over the number of negative eigenvalues of C , each counted with its proper multiplicity. We have the following properties of the index of a real symmetric matrix:

For a nonsingular $(r \times r)$ -matrix T we have

$$(3) \quad \tau(C) = \tau({}^tTCT).$$

Here, as always, we denote by tT the transpose of T . For nonsingular square matrices A, L (with A symmetric) we have

$$(4) \quad \tau \begin{pmatrix} 0 & 0 & L \\ 0 & A & 0 \\ {}^tL & 0 & 0 \end{pmatrix} = \tau \begin{pmatrix} 0 & L \\ {}^tL & 0 \end{pmatrix} + \tau(A) = \tau(A).$$

Here and always we make use of the convention that the index of the empty matrix is zero.

To prove (4) it is enough to show that

$$(5) \quad \tau \begin{pmatrix} 0 & L \\ {}^tL & 0 \end{pmatrix} = 0.$$

In this case, r is even. Put $r = 2\mu$. Obviously, the cone $F(x, x) = 0$ of the symmetric bilinear function $F(x, y)$ belonging to the matrix

$$\begin{pmatrix} 0 & L \\ {}^tL & 0 \end{pmatrix}$$

contains a linear space of dimension μ . Thus $\min(p, q) \geq \mu$. On the other hand, $p + q = 2\mu$. Therefore, $p = q$ and $\tau = 0$.

LEMMA 1. Let C be a real, symmetric, nonsingular matrix of the form

$$C = \begin{pmatrix} 0 & & & L_0 \\ & \ddots & & \\ & & \ddots & \\ L_m & & & * \end{pmatrix}$$

where $L_0, \dots, L_m$ are square matrices (empty matrices are admitted) and where L_i is the transpose of L_{m-i} . Then

$$\tau(C) = \tau \begin{pmatrix} 0 & & & L_0 \\ & \ddots & & \\ & & \ddots & \\ L_m & & & 0 \end{pmatrix} = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \tau(L_n), & \text{if } m = 2n. \end{cases}$$

PROOF. We put

$$(6) \quad C_\lambda = \begin{pmatrix} 0 & & & L_0 \\ & \ddots & & \\ & & \ddots & \\ L_m & & & \lambda * \end{pmatrix}, \quad 0 \leq \lambda \leq 1.$$

Since $\det(C_\lambda) = \pm \prod_{i=0}^m \det(L_i) \neq 0$, the index $\tau(C_\lambda)$ is obviously independent of λ , so that $\tau(C) = \tau(C_1) = \tau(C_0)$. By (4) we have $\tau(C_0) = 0$ resp. $\tau(C_0) = \tau(L_n)$, q.e.d.

LEMMA 2. Let A and B be two square matrices, which are either both symmetric or both skew-symmetric. Then their tensor product $A \otimes B$ is symmetric, and

$$(7) \quad \tau(A \otimes B) = \tau(A)\tau(B) \text{ or } 0,$$

according as both A and B are symmetric or skew-symmetric.

Suppose first that A and B are both symmetric. Let $\alpha_i > 0$, $\alpha_j < 0$, $1 \leq i \leq p$, $p+1 \leq j \leq p+q$, be the nonzero eigenvalues of A and $\beta_k > 0$, $\beta_l < 0$, $1 \leq k \leq p'$, $p'+1 \leq l \leq p'+q'$ be the nonzero eigenvalues of B . Then the nonzero eigenvalues of $A \otimes B$ are $\alpha_u \beta_t$, $1 \leq u \leq p+q$, $1 \leq t \leq p'+q'$. It follows that

$$\tau(A \otimes B) = pp' + qq' - pq' - p'q = \tau(A)\tau(B).$$

Now let A and B be both skew-symmetric. By applying (3) to the matrix $C = A \otimes B$ we can suppose that A and B are both of the form

$$\begin{pmatrix} A_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & A_n \\ & & & & 0 \end{pmatrix}$$

where each A_i is a 2×2 block:

$$A_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J.$$

Since

$$\tau\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \tau\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} = 0,$$

we have $\tau(A \otimes B) = 0$.

2. Poincaré rings. We consider a graded ring A with the following properties:

(1) In the direct sum decomposition

$$A = \sum_{0 \leq r < \infty} A^r$$

of A into the subgroups of its homogeneous elements, each A^r is a real vector space of finite dimension. There exists an n with $A^r = 0$ for $r > n$ and with $\dim A^n = 1$.

(2) If $x \in A^i$, $y \in A^j$ then $xy \in A^{i+j}$ and

$$xy = (-1)^{ij}yx.$$

Let $\xi \neq 0$ be a base element of A^n . Relative to ξ we define a bilinear pairing $\langle x, y \rangle$ of A^r and A^{n-r} into the real field by the equation

$$\langle x, y \rangle \xi = xy, \quad x \in A^r, y \in A^{n-r}.$$

Let i_{n-r} be the linear mapping of A^{n-r} into $(A^r)^*$, the dual vector space of A^r , which assigns to $y \in A^{n-r}$ the linear function $\langle x, y \rangle$ on A^r ($x \in A^r$).

A graded ring A is called a Poincaré ring if it satisfies (1), (2) and has moreover the following property:

(3) The mapping i_{n-r} is a bijection of A^{n-r} onto $(A^r)^*$.

A consequence of (3) is

$$\dim A^r = \dim A^{n-r}, \quad 0 \leq r \leq n.$$

The cohomology ring of a compact orientable manifold is a Poincaré ring.

A differentiation in a Poincaré ring A is a linear endomorphism $d: A \rightarrow A$, satisfying the following conditions:

(α) $dA^r \subset A^{r+1}$;

(β) $dd = 0$;

(γ) $d(xy) = (dx)y + (-1)^r x(dy)$, if $x \in A^r$;

(δ) $dA^{n-1} = 0$.

As is well known, such a differentiation defines a derived ring $A' = d^{-1}(0)/dA$. If we put $A'^r = d^{-1}(0) \cap A^r/dA^{r-1}$, we have the direct sum decomposition

$$A' = \sum_{0 \leq r \leq n} A'^r,$$

and A' is a graded ring. It is easy to verify that, if $x' \in A'^i$, $y' \in A'^j$, then $x'y' \in A'^{i+j}$, and

$$x'y' = (-1)^{ij}y'x'.$$

From the property (δ) of d we have $\dim A'^n = 1$. Thus A' satisfies (1) and (2) with the same maximal degree n as A . We denote the residue class of ξ in A'^n by ξ' . Relative to ξ' we have the linear mapping

$$i'_{n-r}: A'^{n-r} \rightarrow (A^r)^*.$$

LEMMA 3. *The derived ring of a Poincaré ring with differentiation is a Poincaré ring, i.e. i'_{n-r} is bijective.*

It remains to prove that A' has the property (3) in the definition of a Poincaré ring. Let $x \in A^r$, $y \in A^{n-r-1}$. By property (δ) of d , we have

$$0 = d(xy) = (dx)y + (-1)^r x(dy).$$

This gives

$$(8) \quad \langle dx, y \rangle = (-1)^{r-1} \langle x, dy \rangle,$$

a relation which is independent of the choice of ξ . This relation is equivalent to saying that the following diagram is commutative:

$$\begin{array}{ccccc} A^{n-r-1} & \xrightarrow{d} & A^{n-r} & \xrightarrow{d} & A^{n-r+1} \\ \downarrow i_{n-r-1} & & \downarrow i_{n-r} & & \downarrow i_{n-r+1} \\ (A^{r+1})^* & \xrightarrow{(-1)^{r-1}({}^t d)} & (A^r)^* & \xrightarrow{(-1)^r({}^t d)} & (A^{r-1})^* \end{array}$$

where $(A^r)^*$ is the dual space of A^r , and ${}^t d$ is the dual homomorphism of d . We have the canonical isomorphism

$$(A^r)^* \cong {}^t d^{-1}(0) \cap (A^r)^* / {}^t d(A^{r+1})^*.$$

The above diagram shows that i_{n-r} induces an isomorphism, namely i'_{n-r} , of A'^{n-r} onto $(A^r)^*$. It follows that A'^r and A'^{n-r} are dually paired into the real field relative to the element $\xi' \in A'^n$, which is the residue class of ξ .

In analogy with the index of an oriented manifold we can define the index $\tau_{\xi}(A)$ of our Poincaré ring A relative to ξ . It is to be zero, if $n \equiv 0, \text{ mod } 4$. If $n = 4k$, $\tau_{\xi}(A)$ is to be the index of the bilinear function $\langle x, y \rangle$, $x, y \in A^{2k}$. Obviously, $\tau_{\xi}(A) = \tau_{\xi_1}(A)$, if ξ_1 is a positive multiple of ξ .

LEMMA 4. *In a Poincaré ring A let $\xi \neq 0$ be a base of A^n , and let $\xi' \in A'^n$ be the residue class which contains ξ . Then $\tau_{\xi'}(A') = \tau_{\xi}(A)$.*

It is only necessary to prove the lemma for the case $n = 4k$. Let $Z^{2k} = d^{-1}(0) \cap A^{2k}$, $B^{2k} = dA^{2k-1}$, and let a, b, c be the respective dimensions of A^{2k} , B^{2k} , Z^{2k} . It follows immediately from (8) that each of the two spaces B^{2k} and Z^{2k} is the orthogonal of the other with respect to the symmetric form $\langle x, y \rangle$ of A^{2k} , whence $a = b + c$. We have $B^{2k} \subset Z^{2k} \subset A^{2k}$. If e_i is a base of A^{2k} such that $e_i \in B^{2k}$ for $1 \leq i \leq b$ and $e_i \in Z^{2k}$ for $b+1 \leq i \leq c$, the matrix $(\langle e_i, e_j \rangle)$ has then the form

$$\begin{pmatrix} 0 & 0 & L \\ 0 & Q & * \\ {}^tL & * & * \end{pmatrix},$$

where L and Q are square nonsingular matrices, of orders b and $c-b$ respectively. Its index is $\tau_{\xi}(A)$, while $\tau(Q)$ is $\tau_{\xi'}(A')$. By Lemma 1, we get therefore $\tau_{\xi'}(A') = \tau_{\xi}(A)$, as contended.

3. Proof of the theorem. It suffices to prove the theorem (see Introduction) for the case $\dim E = 4k$, which we suppose from now on. We consider the cohomology spectral sequence $E_r^{p,q}$, $2 \leq r \leq \infty$, of the bundle $E \rightarrow B$, with the real field as the coefficient field. Let

$$E_r^s = \sum_{p+q=s} E_r^{p,q}, \quad E_r = \sum_{0 \leq s} E_r^s, \quad 2 \leq r \leq \infty.$$

Each E_r is a graded ring, satisfying $E_r^s E_r^{s'} \subset E_r^{s+s'}$ and also $E_r^{p,q} E_r^{p',q'} \subset E_r^{p+p', q+q'}$. It has a differentiation d_r , such that E_{r+1} is the derived ring of E_r . In our case d_r is trivial for sufficiently large r and E_{∞} , or E_r for r sufficiently large, is the graded ring belonging to a certain filtration of the cohomology ring of the manifold E . The term E_2 of the spectral sequence is by hypothesis (2) of our theorem isomorphic to $H^*(B, H^*(F)) = H^*(B) \otimes H^*(F)$, such that

$$E_2^{p,q} \cong H^p(B, H^q(F)) \cong H^p(B) \otimes H^q(F).$$

If we identify $E_2^{p,q}$ with $H^p(B) \otimes H^q(F)$ under this isomorphism, the multiplication in E_2 is given by

$$(b \otimes f)(b' \otimes f') = (-1)^{p'q}(b \cup b') \otimes (f \cup f'),$$

$$b \in H^p(B), \quad b' \in H^{p'}(B), \quad f \in H^q(F), \quad f' \in H^{q'}(F).$$

Let $m = \dim F$, so that $\dim B = 4k - m$. Since B and F are manifolds, E_2 is a Poincaré ring with respect to the grading

$$E_2 = \sum_{0 \leq s < \infty} E_2^s \quad (E_2^s = 0 \text{ for } s > 4k, E_2^{4k} = E_2^{4k-m, m}).$$

The ring E_2 is isomorphic to the cohomology ring of $B \times F$.

The orientations of B, F define a generator $\xi_2 = \xi_B \otimes \xi_F$ of E_2^{4k} . Here ξ_B (resp. ξ_F) denotes the generator of $H^{4k-m}(B)$ (resp. $H^m(F)$) belonging to the orientation of B (resp. F). We wish to prove that

$$\tau_{\xi_2}(E_2) = \tau(B) \cdot \tau(F).$$

We have

$$(9) \quad E_2^{2k} = E_2^{2k, 0} + E_2^{2k-1, 1} + \cdots + E_2^{2k-m, m}.$$

Here some of the $E_2^{p, q}$ might vanish, in particular $E_2^{p, q} = 0$ if $p < 0$. Clearly, for $x \in E_2^{2k-q, q}$ and $y \in E_2^{2k-q', q'}$ we have $xy = 0$ unless

$$q + q' = m.$$

By Poincaré duality in B and F , we have

$$\dim E_2^{2k-q, q} = \dim E_2^{2k-m+q, m-q}.$$

Therefore, the symmetric matrix, which defines the bilinear symmetric function over E_2^{2k} , is, when written in blocks relative to the direct sum decomposition (9), of the form

$$\begin{pmatrix} 0 & & & L_0 \\ & \ddots & & \\ & & \ddots & \\ L_m & & & 0 \end{pmatrix}$$

where the L_i are nonsingular square matrices, such that L_i is the transpose of L_{m-i} . By Lemma 1 we obtain

$$\tau_{\xi_2}(E_2) = 0 \text{ if } m \text{ is odd,} \quad \tau_{\xi_2}(E_2) = \tau(L_{m/2}) \text{ if } m \text{ is even.}$$

In the first case the equation $\tau_{\xi_2}(E_2) = \tau(B)\tau(F)$ is proved, since $\tau_{\xi_2}(E_2) = \tau(F) = 0$. In the latter case we have

$$E_2^{2k-m/2, m/2} = H^{2k-m/2}(B) \otimes H^{m/2}(F),$$

and it is clear that up to the sign $(-1)^{m/2}$ the matrix $L_{m/2}$ is the tensor product of the two matrices defining the bilinear forms of B and F . If $m/2$ is odd, both matrices in this tensor product are skew-symmetric, and we have, by Lemma 2, $\tau(L_{m/2})=0$; on the other hand we have $\tau(B)\tau(F)=0$, since $\dim F \not\equiv 0 \pmod{4}$ and thus by definition $\tau(F)=0$. If $m/2$ is even, that is, if $m \equiv 0 \pmod{4}$, both matrices are symmetric, and Lemma 2 gives: $\tau(L_{m/2})=\tau(B)\tau(F)$. Combining all cases, we get the formula

$$(10) \quad \tau_{\xi_2}(E_2) = \tau(B)\tau(F)$$

in full generality.

The differentiation d_2 of E_2 satisfies the conditions of a differentiation in a Poincaré ring given in §2. In fact, $\dim E_\infty^{4k}=1$, since E is a manifold of dimension $4k$. Therefore, $\dim E_r^{4k}=1$ for $2 \leq r$. Thus d_2 annihilates E_2^{4k-1} ; more generally d_r annihilates E_r^{4k-1} . It follows by Lemma 3 that E_3 is a Poincaré ring. It has d_3 as differentiation and therefore E_4 is a Poincaré ring etc. Finally, E_∞ is a Poincaré ring. By Lemma 4 and (10) we get

$$\tau(B)\tau(F) = \tau_{\xi_2}(E_2) = \tau_{\xi_3}(E_3) = \cdots = \tau_{\xi_\infty}(E_\infty),$$

where ξ_r (resp. ξ_∞) is the image of ξ_2 in E_r (resp. E_∞).

It remains to prove that $\tau_{\xi_\infty}(E_\infty)=\tau(E)$. The cohomology ring $H^*(E)$ is filtered:

$$(11) \quad \begin{aligned} H^*(E) &= D^0 \supset D^1 \supset \cdots \supset D^p \supset D^{p+1} \supset \cdots, & \cap D^p &= 0, \\ D^{p,q} &= D^p \cap H^{p+q}(E), \\ D^{p,q} \cdot D^{p',q'} &\subset D^{p+p',q+q'}. \end{aligned}$$

We have the filtration

$$H^r(E) = D^{0,r} \supset D^{1,r-1} \supset \cdots \supset D^{r,0} \supset D^{r+1,-1} = 0$$

and the canonical isomorphism

$$(12) \quad D^{p,q}/D^{p+1,q-1} \cong E_\infty^{p,q}.$$

The ring structure of E_∞ is induced by that of $H^*(E)$ by the canonical homomorphisms $D^{p,q} \rightarrow E_\infty^{p,q}$ (see (12) and (11)). Since $E_\infty^{4k} = E_\infty^{4k-m,m}$, (where $m = \dim F$), we have

$$(13) \quad H^{4k}(E) = D^{4k-m,m} \cong E_\infty^{4k-m,m}$$

and

$$(14) \quad D^{4k-i,i} = 0 \quad \text{for } i < m.$$

Earlier we have chosen a generator $\xi_\infty \in E_\infty^{4k}$. Under the canonical isomorphism (13) ξ_∞ goes over in the generator ξ_E of $H^{4k}(E)$ belonging to the orientation of E generated by the given orientations of B and F in this order.² We now consider the bilinear symmetric function $\langle x, y \rangle$ over $H^{2k}(E)$ relative to ξ_E . Choose a direct sum decomposition of $H^{2k}(E)$ in linear subspaces,

$$(15) \quad H^{2k}(E) = V_0 + V_1 + V_2 + \cdots + V_m$$

such that

$$\sum_{j=0}^q V_j = D^{2k-q,q} \quad (0 \leq q \leq m).$$

Here we use that $D^{2k-s,s} = D^{2k-m,m}$ for $s > m$. By (11) and (14) we have

$$(16) \quad \langle x, y \rangle = 0 \quad \text{for } x \in V_i, y \in V_j \text{ and } i + j < m,$$

and moreover by (13)

$$(17) \quad \langle x, y \rangle = \langle \bar{x}, \bar{y} \rangle, \quad \text{for } x \in V_i, y \in V_j \text{ and } i + j = m,$$

where $\bar{x}$ (resp. $\bar{y}$) denotes the image (see (12)) of x (resp. y) in E_∞^{2k-i} (resp. E_∞^{2k-j}) and where on the right side of this equation stands the symmetric bilinear form over E_∞^{2k} relative to ξ_∞ . Since $\langle \bar{x}, \bar{y} \rangle = 0$ for $\bar{x} \in E_\infty^{2k-q,q}$, $\bar{y} \in E_\infty^{2k-q',q'}$, unless $q + q' = m$, and since E_∞ is a Poincaré algebra, we can conclude

$$(18) \quad \dim E_\infty^{2k-q,q} = \dim E_\infty^{2k-m+q,m-q}.$$

The preceding remarks, in particular (16), (17), (18), imply: The matrix of the symmetric bilinear function over $H^{2k}(E)$ relative to ξ_E can be written in blocks with respect to the direct sum decomposition (15) in the form

$$\begin{pmatrix} 0 & & & & L_0 \\ & & & & \\ & & L_1 & & \\ & & \cdot & & \\ & & \cdot & & \\ L_m & & & & * \end{pmatrix}$$

² This is easy to see when E is a trivial bundle, in which case it is almost the definition of the orientation of a product of manifolds. The general case can be reduced to this one by comparing the spectral sequence of E to that of the bundle induced by E on an open cell of the base, the cohomology being taken with compact carriers.

where the L_i are nonsingular square matrices and where L_i is the transpose of L_{m-i} . Moreover,

$$\begin{pmatrix} & & & L_0 \\ 0 & & & \\ & \ddots & & \\ & & \ddots & \\ L_m & & & 0 \end{pmatrix}$$

is the matrix of the symmetric bilinear function over E_∞^{2k} relative to ξ_∞ . By Lemma 1 we have $\tau(E) = \tau_{\xi_\infty}(E_\infty)$. This concludes the proof of our theorem.

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THE PERIPHERAL CHARACTER OF CENTRAL ELEMENTS OF A LATTICE¹

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A lattice being a Hausdorff space together with a pair of continuous lattice operations ($\wedge$ and $\vee$) the content of this note is best exhibited by quoting a corollary to our theorem: *If a compact connected lattice is (topologically) situated in Euclidean n -space then its center is contained in its boundary.* Thus, far from being "centrally located," the central elements are "peripheral."

The above is a consequence (see [3, p. 273]) of the

THEOREM. *If L is a compact connected lattice, if R is an (n, G) -rim [3] for L and if (i) a is central [1, p. 27] or if (ii) L is modular and a is complemented then $a \in R$.*

PROOF. The procedure is to introduce an appropriate multiplication into L so that L is a semigroup, to show that L is not simple (in the semigroup sense [3]) and that a is a left unit. Since L is compact it has a zero and unit, 0 and 1, as is well-known. Indeed, the set $\bigcap \{x \vee L \mid x \in L\}$ is easily seen to consist of exactly one element, namely 1. If $a = 1$ then the hypotheses of Theorem 1 of [3] are fulfilled using the multiplication $(x, y) \rightarrow x \wedge y$ so that 1 being a unit for the multiplication, $1 \in R$. If $a \neq 1$ let $x \cdot y = (a' \wedge x) \vee y$, a' being a

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