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A NOTE ON THE LAW OF LARGE NUMBERS

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Let $\{X_k\}$, $k=1, 2, \dots$, be a sequence of independent random variables with mean $EX_k=0$, $k=1, 2, \dots$. Let $F_k(x)$ be the distribution function of X_k , $k=1, 2, \dots$. We list certain conditions which may or may not obtain for such a sequence:

- (i) $\sum_{k=1}^n \int_{|x| \geq n} dF_k \rightarrow 0$ as $n \rightarrow \infty$;
- (ii) $\frac{1}{n} \sum_{k=1}^n \int_{|x| < n} x dF_k \rightarrow 0$ as $n \rightarrow \infty$;
- (iii) $\frac{1}{n^2} \sum_{k=1}^n \int_{|x| < n} x^2 dF_k \rightarrow 0$ as $n \rightarrow \infty$;
- (iv) $\frac{1}{n^2} \sum_{k=1}^n \left\{ \int_{|x| < n} x^2 dF_k - \left(\int_{|x| < n} x dF_k \right)^2 \right\} \rightarrow 0$ as $n \rightarrow \infty$.

Kolmogorov proved in [1] that (i), (ii) and (iv) together are necessary and sufficient conditions for the classical weak law of large numbers. In [1, Satz XI] the statement is also made (without proof) that (i), (ii) and (iii) together are necessary and sufficient conditions for the classical weak law. This statement has appeared more recently in various texts and monographs. We show that this statement is in-

correct by supplying an example for which conditions (i), (ii) and (iv) hold but for which condition (iii) does not hold. At the same time we emphasize that this emendation is minor and that the central, fundamental result of [1] remains unaltered.

In [2] Feller proved that (i), (ii) and (iii) are indeed necessary and sufficient conditions for the classical weak law in the presence of a certain mild condition of uniformity. This condition of uniformity is, of course, not satisfied by our example. The phenomenon illustrated in our example is essentially the same as that illustrated in a similar example provided by Feller [3, p. 557, under e] for the more complicated case of the central limit theorem.

We define a sequence of random variables $\{X_k\}$, $k=1, 2, \dots$, by specifying: (1) X_k takes the value $(-1)^{k+1}k^{1/2}$ with probability $k^2/(k^2+k^{1/2})$ and the value $(-1)^k k^2$ with probability $k^{1/2}/(k^2+k^{1/2})$, $k=1, 2, \dots$; and (2) the $\{X_k\}$ are independent.

Observing that $EX_k=0$, we compute the limits for conditions (i), (ii), (iii) and (iv).

$$(i) \quad \sum_{k=1}^n \int_{|x| \geq n} dF_k \leq \sum_{k=[n^{1/2}]}^{\infty} \frac{k^{1/2}}{k^2 + k^{1/2}} \rightarrow 0$$

and condition (i) holds.

$$(ii) \quad \left| \frac{1}{n} \sum_{k=1}^n \int_{|x| < n} x dF_k \right| \leq \frac{1}{2n} \left\{ \int_{n^{1/2}-2}^{n^{1/2}+2} x^{1/2} dx + \int_{n-2}^{n+2} x^{1/2} dx \right\} \rightarrow 0$$

and condition (ii) holds.

$$(iii) \quad \frac{1}{n^2} \sum_{k=1}^n \int_{|x| < n} x^2 dF_k \geq \frac{1}{n^2} \sum_{k=1}^{[n^{1/2}]} \frac{k^{9/2}}{k^2 + k^{1/2}} + \frac{1}{n^2} \sum_{k=1}^n \frac{k^3}{k^2 + k^{1/2}} \rightarrow \frac{1}{2}$$

and condition (iii) does not hold.

$$(iv) \quad \frac{1}{n^2} \sum_{k=1}^n \left\{ \int_{|x| < n} x^2 dF_k - \left(\int_{|x| < n} x dF_k \right)^2 \right\} \\ \leq \frac{1}{n^2} \sum_{k=1}^{[n^{1/2}]} \frac{k^{9/2} + k^2}{k^2 + k^{1/2}} \rightarrow 0 \quad \text{and condition (iv) holds.}$$

This completes the argument.

Let $Y_k = (-1)^k X_k$. The sequence $\{Y_k\}$ is of independent interest, in that it is a sequence of independent variables which are expectation centered and whose arithmetic means $S_n = (1/n) \sum_{k=1}^n Y_k$, $n=1, 2, \dots$, are stable but not normally stable. (For these notions see [4].) The $\{S_n\}$ are not normally stable since condition (ii) does not hold for the $\{Y_k\}$ (where we now take F_k to be the distribution

function of Y_k , $k=1, 2, \dots$). The $\{S_n\}$ are stable since the following sufficient conditions for stability hold (see [5, §22]; m_k is the median of Y_k , $k=1, 2, \dots$):

$$(a) \quad \sum_{k=1}^n \int_{|x-m_k| \geq n} dF_k \rightarrow 0 \quad \text{as } n \rightarrow \infty;$$

$$(b) \quad \frac{1}{n^2} \sum_{k=1}^n \int_{|x-m_k| < n} (x - m_k)^2 dF_k \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

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