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ON THE IDENTITIES OF CERTAIN ALGEBRAS

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Introduction. One basic problem in the study of finite algebras is concerned with the existence of a finite basis for the identities of the algebra, i.e., a finite set of identities from which *all* the identities of the algebra are logical consequences. That the identities of a finite algebra need not have a finite basis (in the above sense) has already been observed by Lyndon [**3**]. This leads naturally to the following question: are there certain classes of finite algebras the identities of which possess a finite basis? We shall answer this question in the affirmative. Indeed, the main result of this paper is the following

THEOREM. *A functionally strictly complete algebra which contains more than one element has a finite basis for its identities.*

1. **Preliminary concepts.** In this section, we shall review some basic concepts and definitions, all of which are to be found in Foster [**1**; **2**].

Let $\mathfrak{A}=(A, \rho, \cdots)$ be a universal algebra with primitive operations $\rho, \cdots$. Let $A=\{\cdots, x, \cdots\}$.

A (k-ary) $\mathfrak{A}$-*function* $f(x_1, \cdots, x_k)$ is a composition, via the primitive operations, of indeterminate symbols $x_1, \cdots, x_k$ over the set A together with a (possibly empty) set of constants[1] ($=$fixed$\in A$).

An $\mathfrak{A}$-function is called *strict* if it involves no constants.

In an obvious way each $\mathfrak{A}$-function $f(x_1, \cdots, x_k)$ represents (or has associated with it) a mapping of the set A^k into A, where of course different $\mathfrak{A}$-functions need not represent different such mappings. If $\mathfrak{A}$-functions $f(x, \cdots)$ and $g(x, \cdots)$ represent the same mapping we

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[1] In more uniform terminology a constant may also be defined as a "0-ary" function.

speak of an $\mathfrak{A}$-*identity*, $f(x, \cdots)=g(x, \cdots)$. If both f and g are strict $\mathfrak{A}$-functions we speak of a *strict* $\mathfrak{A}$-*identity*.

$\mathfrak{A}$ is *finite*, of *order* n, if A is a class of n elements.

$\mathfrak{A}$ is said to be (*functionally*) *complete*—respectively (*functionally*) *strictly complete*—if A is *finite* and if each mapping of the set $A\times \cdots \times A$ into A may be expressed as some $\mathfrak{A}$-*function*—respectively as some *strict* $\mathfrak{A}$-*function*. For examples and criteria of functional completeness see [**1**; **2**].

2. The finite basis theorem. In preparation for this theorem, we recall the following result which was proved in [**2**] and which we shall state as a lemma.

LEMMA 1. *Let* $\mathfrak{U}$, $\overline{\mathfrak{U}}$ *be any universal algebras of the same species and each containing at least two elements, and where*

(a) $\mathfrak{U}$ *is finite and functionally strictly complete.*

(b) *Each strict identity of* $\mathfrak{U}$ *is also an identity of* $\overline{\mathfrak{U}}$.

Then

(i) $\overline{\mathfrak{U}}$ *is isomorphic with a subdirect sum of* $\mathfrak{U}$.

(ii) $\mathfrak{U}$ *and* $\overline{\mathfrak{U}}$ *satisfy precisely the same strict identities.*

This is an immediate combination of Theorem 9.1, Theorem 9.2 and Theorem 6.2 of [**2**].

Although in the above lemma we assumed that *each* strict identity of $\mathfrak{U}$ is also an identity of $\overline{\mathfrak{U}}$, in the proof of this lemma only a *finite* number of strict $\mathfrak{U}$-identities were assumed to be $\overline{\mathfrak{U}}$-identities (see [**1**; **2**]). Let this *finite* set of strict $\mathfrak{U}$-identities be denoted by I. It is now fairly evident that the above lemma can be strengthened as follows:

LEMMA 1′. *Same as Lemma* 1 *except that hypothesis* (b) *is now replaced by the weaker hypothesis*

(b′) *Each strict identity of* $\mathfrak{U}$ *in the set* I *is also an identity of* $\overline{\mathfrak{U}}$.

We are now in a position to prove the following important

THEOREM 1. *Let* $\mathfrak{U}$ *be a functionally strictly complete algebra of order* $n\geqq 2$. *Then the strict identities of* $\mathfrak{U}$ *have the above finite set* I *of strict* $\mathfrak{U}$-*identities as a finite basis.*

PROOF. Let $\mathfrak{B}$ be any algebra which satisfies set I and let L be any strict identity of $\mathfrak{U}$. Theorem 1 will be proved by showing that L is also a strict identity of $\mathfrak{B}$. But this follows, since if $\mathfrak{B}$ is a one-element algebra, $\mathfrak{B}$ obviously satisfies L, while if $\mathfrak{B}$ has order greater than 1, $\mathfrak{B}$ again satisfies L by Lemma 1′ (conclusion (ii)). This proves the theorem.

We shall conclude the paper by referring to Lyndon's example [**3**]. It is of course to be noted that his algebra is not strictly functionally complete. This is easily seen from Theorem 1 above, and is indeed clear by inspection.

In conclusion, I wish to express my gratitude and indebtedness to Professor A. L. Foster for his generous counsel.

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