QUASICREATIVE SETS¹

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The creative sets of Post [4] are the recursively enumerable sets whose complements are productive in the sense of Dekker [1]. In [1], Dekker introduces the class of semiproductive sets, which properly includes the class of productive sets. If we define a set to be *semicreative* if it is recursively enumerable and has a semiproductive complement, we obtain a natural generalization of creative sets.

In this paper, we show that this generalization actually introduces some new sets. We do this by introducing an intermediate class of recursively enumerable sets, and giving an example of a noncreative set in this class. This example also gives two nonseparable recursively enumerable sets which are not effectively nonseparable.

We use the notation of [1]. In particular, ω_n is the n th recursively enumerable set in a fixed effective enumeration. If $n+1 = 2^{a_1} + \dots + 2^{a_k}$ with $a_1 > \dots > a_k$, we set $\alpha_n = \{a_1, \dots, a_k\}$. Thus every nonempty finite set is an α_n for just one n .

A set A is *quasiprimitive* if there is a partial recursive function f such that if $\omega_n \subset A$, then $f(n)$ is defined, $\alpha_{f(n)} \subset A$, and $\alpha_{f(n)} \not\subset \omega_n$. We then call f a *quasiprimitive function* of A . A set is *quasicreative* if it is recursively enumerable and has a quasiprimitive complement. A quasicreative set is semicreative [1, Proposition G]; the author does not know if the converse holds.

THEOREM 1.² *A set A is quasiprimitive if and only if there is a recursive function f such that:*

$$(1) \quad \alpha_{f(n)} \subset A \leftrightarrow \alpha_{f(n)} \not\subset \omega_n$$

for all n .

PROOF. If such an f exists, it is a quasiprimitive function of A . For suppose $\omega_n \subset A$. If $\alpha_{f(n)} \not\subset A$, $\alpha_{f(n)} \subset \omega_n \subset A$ by (1); and this is a contradiction. Hence $\alpha_{f(n)} \subset A$ and $\alpha_{f(n)} \not\subset \omega_n$.

Now suppose A is quasiprimitive with quasiprimitive function g . Let k be a recursive function such that:

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² The proofs of Theorems 1 and 2 are similar to unpublished proofs given by Myhill of corresponding theorems on productive sets.

$$\omega_{k(n)} = \phi \quad \text{if } g(n) \text{ is undefined,}$$

$$\omega_{k(n)} = \omega_n \quad \text{if } g(n) \text{ defined.}$$

If $g(n)$ is undefined, $\omega_{k(n)} = \phi \subset A$; so $gk(n)$ is defined. Hence there is a recursive function h such that $h(n) = g(n)$ or $h(n) = gk(n)$ for all n . If $\omega_n \subset A$, $\omega_{k(n)} = \omega_n$; it follows that h is a quasiprimitive function of A .

Now let:

$$\begin{aligned} \beta_k^n &= \alpha_{h(k)} \quad \text{if } \alpha_{h(k)} \subset \omega_n, \\ &= \phi \quad \text{if } \alpha_{h(k)} \not\subset \omega_n. \end{aligned}$$

Then $x \in \beta_k^n$ is recursively enumerable in (x, n, k) . By a theorem of Myhill [3, Theorem 8], there is a recursive function b such that $\omega_{b(n)} = \beta_{b(n)}^n$ for all n . Set $f = hb$. Then:

$$(2) \quad \begin{aligned} \omega_{b(n)} &= \alpha_{f(n)} \quad \text{if } \alpha_{f(n)} \subset \omega_n, \\ &= \phi \quad \text{if } \alpha_{f(n)} \not\subset \omega_n. \end{aligned}$$

To prove (1), suppose $\alpha_{f(n)} \subset A$. By (2), $\omega_{b(n)} \subset A$; so $\alpha_{f(n)} = \alpha_{hb(n)} \neq \omega_{b(n)}$. By (2), $\alpha_{f(n)} \not\subset \omega_n$. Conversely, if $\alpha_{f(n)} \not\subset \omega_n$, then $\omega_{b(n)} = \phi \subset A$ by (2), so $\alpha_{f(n)} = \alpha_{hb(n)} \subset A$.

THEOREM 2.2 *If A is quasiprimitive, then every recursively enumerable set is reducible to A by truth tables.*

PROOF. Let f be as in Theorem 1. Let B be recursively enumerable. Let:

$$\begin{aligned} \beta_n^k &= \alpha_{f(k)} \quad \text{if } n \in B, \\ &= \phi \quad \text{if } n \notin B. \end{aligned}$$

As above, there is a recursive function b such that $\omega_{b(n)} = \beta_n^{b(n)}$. Let $g = fb$; then:

$$\begin{aligned} \omega_{b(n)} &= \alpha_{g(n)} \quad \text{if } n \in B, \\ &= \phi \quad \text{if } n \notin B. \end{aligned}$$

Then, since $\alpha_{g(n)} \neq \phi$:

$$\alpha_{g(n)} \not\subset \omega_{b(n)} \leftrightarrow n \notin B.$$

By choice of f :

$$\alpha_{g(n)} \subset A \leftrightarrow \alpha_{g(n)} \not\subset \omega_{b(n)}.$$

Hence:

$$\alpha_{g(n)} \subset A \leftrightarrow n \notin B.$$

The theorem follows.

COROLLARY 1. *If A is quasicreative, then every recursively enumerable set is reducible to A by truth tables.*

COROLLARY 2. *Any two quasicreative sets are reducible to each other by truth tables.*

We now construct two disjoint recursively enumerable sets A and B such that A is quasicreative and $A \cup B$ is simple.

Let f be a recursive function such that:

$$\begin{aligned}\alpha_{f(0)} &= \{0, 1, 2, 3, 4\}, \\ \alpha_{f(n)} &= \{2^{n+1} + 1, 2^{n+1} + 2, \dots, 2^{n+2}\}.\end{aligned}$$

In [4], Post constructed a simple set S such that $\alpha_{f(n)} \not\subseteq S$ for all n . Let T be the union of S and all $\alpha_{f(n)}$ such that $\alpha_{f(n)} \subset \omega_n$. Clearly T is recursively enumerable. Let g be a one-one recursive function enumerating T . A is to consist of all elements $g(k)$ of T with the following property: if $g(k) \in \alpha_{f(n)}$, then $\alpha_{f(n)} \subset \{g(0), \dots, g(k)\}$. We let $B = T - A$.

To see that A and B are recursively enumerable, let $P(k, x)$ mean that $\alpha_{f(n)} \subset \{g(0), \dots, g(k)\}$, where n is chosen so that $x \in \alpha_{f(n)}$. Then $P(k, x)$ is recursive, and:

$$\begin{aligned}x \in A &\leftrightarrow (\exists k)(x = g(k) \cdot P(k, x)), \\ x \in B &\leftrightarrow (\exists k)(x = g(k) \cdot \sim P(k, x)).\end{aligned}$$

Now $A \cap \alpha_{f(n)} \neq \emptyset$ if and only if $\alpha_{f(n)} \subset T$, hence if and only if $\alpha_{f(n)} \subset \omega_n$. Then by Theorem 1, A' is quasiprimitive; so A is quasicreative. Hence A' is not recursively enumerable; so $T' = A' - B$ cannot be finite. Since T includes the simple set S , it follows that $T = A \cup B$ is simple.

It follows that A is not creative. For if it were, we would obtain an infinite recursively enumerable subset of T' by effectively finding x_0 not in $A \cup B$, x_1 not in $A \cup (B \cup \{x_0\})$, etc.

Any disjoint recursively enumerable sets A and B are nonseparable if there is no recursive set C such that $A \subset C$ and $B \subset C'$. They are effectively nonseparable if there is an effective process by which, given any disjoint recursively enumerable sets C and D with $A \subset C$ and $B \subset D$, we can effectively find an $x \notin C \cup D$.

We note that the A and B constructed above are nonseparable, but not effectively nonseparable.³ For suppose C is recursive, $A \subset C$, and

³ This answers a question of Uspenskij [5]. Other examples have previously been given by Tennenbaum (unpublished) and Muchnik [2].

$B \subset C'$. As C' is a recursively enumerable subset of the quasiprodutive set A' , there is a recursively enumerable set D such that $D \subset A'$ and $D \cap C = D - C'$ is infinite [1, Proposition H]. Then $D \cap C \subset T'$, contradicting the simplicity of T . On the other hand, if A and B were effectively nonseparable, we could find an infinite recursively enumerable subset of T' by choosing x_0 not in $A \cup B$, x_1 not in $A \cup (B \cup \{x_0\})$, etc.

Finally, our example shows that truth tables cannot be replaced by bounded truth tables in Theorem 2 and its corollaries. This follows from the following strengthening of a result of Post, which states that a creative set is not reducible to a simple set by bounded truth tables:

THEOREM 3. *If C is creative, and A and B are disjoint recursively enumerable sets whose union is simple, then C is not reducible to A by bounded truth tables.*

PROOF. We shall explain the necessary modifications in the proof of §7 of [4]. We amend Π_p to correctly designate p elements as belonging to A or to B . Thus I is replaced by I_1, I_2 , where $m_i \in A$ if $i \in I_1$ and $m_i \in B$ if $i \in I_2$; and $I_1 \cup I_2$ has p members. In particular, $m_i \notin A$ for $i \in I_2$. Just as in [4], we may show that this process may be defined for every p , which is clearly impossible.

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