THE WHITEHEAD FIRST LEMMA FOR ALTERNATIVE ALGEBRAS¹

EARL JAY TAFT

1. Introduction. The Whitehead first lemma for associative algebras states that any derivation of a separable algebra into a bimodule is inner. In [6], Schafer has proved the analogue of this result for alternative algebras over a field of characteristic zero. He does this by using the corresponding result of Jacobson, [3], for the Jordan algebra and Jordan module formed from an alternative algebra and bimodule. Recent structure theory, due to Jacobson [4] indicates that it may be more natural to first prove this kind of result for alternative algebras, and then use this to prove the analogous result for Jordan algebras. In this paper, we prove the Whitehead first lemma for alternative algebras using only the alternative structure theory. Moreover, the result holds for any base field of characteristic not two or three.

The proof, as given in §3, is a reduction to some special cases. Hence we start with a discussion of derivations in Cayley algebras. The numbers in square brackets refer to the bibliography at the end of the text. All algebras in this paper are finite-dimensional over the base field.

2. Derivations in a Cayley algebra. It is assumed that the characteristic of the base field Φ is not two or three. If $\mathfrak{A}$ is a (nonassociative) algebra over Φ , let $\mathfrak{D}(\mathfrak{A})$ denote the Lie algebra of derivations of $\mathfrak{A}$ over Φ . A derivation is *inner* if it is contained in the enveloping Lie algebra of the right and left multiplications by elements of $\mathfrak{A}$ (cf. [5]).

If Ω is a generalized quaternion algebra over Φ , then the generalized Cayley algebra over Φ is $\mathbb{C} = \Omega 1 + \Omega e_4$, with a multiplication given by $(a + be_4)(c + de_4) = (ac + \bar{d}b\alpha_4) + (da + b\bar{c})e_4$, where $\alpha_4 \in \Phi$, $\alpha_4 \neq 0$, $a, b, c, d \in \Omega$. If Ω has a basis $(1, e_1, e_2, e_3)$, then set $e_5 = e_1e_4$, $e_6 = e_4e_2$, and $e_7 = e_3e_4$. Then $1, e_1, e_2, \dots, e_7$ is a basis for $\mathbb{C}$. If $q \rightarrow \bar{q}$ denotes the involution in Ω , then $\mathbb{C}$ has the involution $x = q_1 + q_2e_4 \rightarrow \bar{x} = \bar{q}_1 - q_2e_4$. $\mathbb{C}$ is an alternative algebra, and $\mathbb{C} = \Phi 1 \oplus \mathbb{C}^1$, where $\mathbb{C}^1$ is the space of elements of trace zero. (Trace $x = x + \bar{x}$). Since e_1, e_2, e_4 generate $\mathbb{C}$, any derivation is uniquely determined by its action on e_1, e_2, e_4 . Jacobson has shown, [2], that $\mathbb{C}^1$ is an invariant $\mathfrak{D}(\mathbb{C})$ -

Received by the editors March 14, 1957.

¹ The material in this paper is a portion of a dissertation submitted to Yale University, prepared under the guidance of Professor Nathan Jacobson.

subspace, and that if $e_1 D = \sum_{i=1}^7 \lambda_i e_i$, $e_2 D = \sum_{i=1}^7 \mu_i e_i$, $e_4 D = \sum_{i=1}^7 \gamma_i e_i$, then

$$(1) \quad \begin{aligned} \lambda_1 = \mu_2 = \nu_4 = \mu_1 \alpha_1 + \lambda_2 \alpha_2 &= \nu_1 \alpha_1 + \lambda_4 \alpha_4 = \nu_2 \alpha_2 + \mu_4 \alpha_4 \\ &= \lambda_6 \alpha_4 \alpha_2 + \mu_5 \alpha_1 \alpha_4 - \nu_3 \alpha_1 \alpha_2 = 0. \end{aligned}$$

(Note: $\alpha_1 = e_1^2$, $\alpha_2 = e_2^2$). It follows that $\mathfrak{D}(\mathfrak{E})$ is a fourteen-dimensional Lie algebra. Using (1), we display a basis for $\mathfrak{D}(\mathfrak{E})$, denoted D_1 , D_2 , $\dots$, D_{14} . Only the effect on e_1 , e_2 , e_4 is shown.

	e_1	e_2	e_4
D_1	e_2	$-\alpha_2 \alpha_1^{-1} e_1$	0
D_2	e_3	0	0
D_3	e_4	0	$-\alpha_4 \alpha_1^{-1} e_1$
D_4	e_5	0	0
D_5	e_6	0	$\alpha_4 \alpha_1^{-1} e_3$
D_6	e_7	0	0
D_7	0	e_3	0
D_8	0	e_4	$-\alpha_4 \alpha_2^{-1} e_2$
D_9	0	e_5	$\alpha_4 \alpha_2^{-1} e_3$
D_{10}	0	e_6	0
D_{11}	0	e_7	0
D_{12}	0	0	e_5
D_{13}	0	0	e_6
D_{14}	0	0	e_7

In any alternative algebra $\mathfrak{A}$, the linear transformation

$$(2) \quad D_{x,y} = [R_x, R_y] + [R_x, L_y] + [L_x, L_y], \quad x, y \in \mathfrak{A}$$

is a derivation, where R_x , L_x denote right and left multiplication, respectively, by x in $\mathfrak{A}$ (See [5]). Hence a linear transformation $\sum_i D_{x_i, y_i}$ is an inner derivation in $\mathfrak{A}$. $D_{x,y}$ is a bilinear function of x and y , $D_{x,x} = 0$, $D_{x,y} = -D_{y,x}$, and $D_{1,x} = D_{x,1} = 0$ for 1 an identity of $\mathfrak{A}$. Hence if $(1, e_1, \dots, e_n)$ is a basis for $\mathfrak{A}$, then $\{\sum D_{x_i, y_i}; x_i, y_i \in \mathfrak{A}\}$

has dimension at most $n(n-1)/2$, and is spanned by D_{e_i, e_j} , for $i < j$.

Returning to $\mathfrak{E}$, we now show that the space spanned by these twenty-one inner derivations is of dimension fourteen, and hence all derivations of $\mathfrak{E}$ are inner and have the form $\sum D_{x_i, v_i}$. We choose fourteen D_{e_i, e_j} , tabulate their action of e_1, e_2, e_4 , and then express them in terms of $D_1, D_2, \dots, D_{14}$, noting their linear independence.

	e_1	e_2	e_4
D_{e_1, e_2}	$4\alpha_1 e_2$	$-4\alpha_2 e_1$	$2e_7$
D_{e_4, e_7}	$2\alpha_1 \alpha_4 e_2$	$-2\alpha_2 \alpha_4 e_1$	$4\alpha_4 e_7$
D_{e_1, e_3}	$4\alpha_1 e_3$	0	$-2\alpha_1 e_6$
D_{e_4, e_6}	$-2\alpha_4 e_3$	0	$4\alpha_4 e_6$
D_{e_1, e_4}	$4\alpha_1 e_4$	$-2e_7$	$-4\alpha_1 e_1$
D_{e_2, e_7}	$-2\alpha_1 \alpha_2 e_4$	$4\alpha_2 e_7$	$2\alpha_2 \alpha_4 e_1$
D_{e_1, e_5}	$4\alpha_1 e_5$	$2\alpha_1 e_6$	0
D_{e_2, e_6}	$2\alpha_2 e_5$	$4\alpha_2 e_6$	0
D_{e_1, e_6}	$4\alpha_1 e_6$	$-2\alpha_2 e_5$	$2\alpha_1 e_3$
D_{e_2, e_5}	$-2\alpha_1 e_6$	$4\alpha_2 e_5$	$2\alpha_4 e_3$
D_{e_1, e_7}	$4\alpha_1 e_7$	$2\alpha_1 \alpha_2 e_4$	$-2\alpha_1 \alpha_4 e_2$
D_{e_2, e_4}	$2e_7$	$4\alpha_2 e_4$	$-4\alpha_1 e_2$
D_{e_2, e_3}	0	$4\alpha_2 e_3$	$-2\alpha_2 e_5$
D_{e_4, e_5}	0	$-2\alpha_4 e_3$	$4\alpha_4 e_5$

Also, the following equations hold.

$$\begin{aligned}
 &\begin{cases} D_{e_1, e_2} = 4\alpha_1 D_1 + 2D_{14}, \\ D_{e_4, e_7} = 2\alpha_1 \alpha_4 D_1 + 4\alpha_4 D_{14}, \end{cases} \\
 &\begin{cases} D_{e_1, e_3} = 4\alpha_1 D_2 - 2\alpha_1 D_{13}, \\ D_{e_4, e_6} = -2\alpha_4 D_2 + 4\alpha_4 D_{13}, \end{cases} \\
 &\begin{cases} D_{e_1, e_4} = 4\alpha_1 D_3 - 2D_{11}, \\ D_{e_2, e_7} = -2\alpha_1 \alpha_2 D_3 + 4\alpha_2 D_{11}, \end{cases}
 \end{aligned}$$

$$\begin{cases} D_{e_1, e_5} = 4\alpha_1 D_4 + 2\alpha_1 D_{10}, \\ D_{e_2, e_6} = 2\alpha_2 D_4 + 4\alpha_2 D_{10}, \\ D_{e_1, e_6} = 4\alpha_1 D_5 - 2\alpha_2 D_9, \\ D_{e_2, e_5} = -2\alpha_1 D_5 + 4\alpha_2 D_9, \\ D_{e_1, e_7} = 4\alpha_1 D_6 + 2\alpha_1 \alpha_2 D_8, \\ D_{e_2, e_4} = 2D_6 + 4\alpha_2 D_8, \\ D_{e_2, e_3} = 4\alpha_2 D_7 - 2\alpha_2 D_{12}, \\ D_{e_4, e_5} = -2\alpha_4 D_7 + 4\alpha_4 D_{12}. \end{cases}$$

In the first pair of equations, if D_{e_1, e_5} and D_{e_4, e_7} were dependent then $(4\alpha_1)(4\alpha_4) = 2(2\alpha_1\alpha_4)$. Since $12 \neq 0$, this is a contradiction. Hence the fourteen D_{e_i, e_j} above are linearly independent and form a basis for $\mathfrak{D}(\mathfrak{C})$.

THEOREM 1.² *Let $\mathfrak{C}$ be a Cayley algebra over a field Φ whose characteristic is not two or three. Then all derivations of $\mathfrak{C}$ over Φ are inner, and have the form $\sum_i D_{x_i, y_i}$, for x_i, y_i in $\mathfrak{C}$.*

In the case of characteristic three, the same sort of calculation as above will show that the $\sum D_{x_i, y_i}$ span only seven dimensions. For characteristic two, the $D_{x, y}$ need not be derivations.

3. Whitehead's first lemma.

THEOREM 2. *Let $\mathfrak{A}$ be a separable alternative algebra over a base field Φ whose characteristic is not two or three. Let (R, L) be an alternative representation of $\mathfrak{A}$ acting in the bimodule $\mathfrak{M}$. Let f be a one-cocycle of $\mathfrak{A}$ into $\mathfrak{M}$, i.e., a linear transformation of $\mathfrak{A}$ into $\mathfrak{M}$ satisfying $f(ab) = f(a)b + af(b) = f(a)R_b + f(b)L_a$, for all a, b in $\mathfrak{A}$. Then there exists an element g belonging to the nucleus of $\mathfrak{M}$, and elements x_i in $\mathfrak{A}$, z_i in $\mathfrak{M}$, such that*

$$(3) \quad f(a) = [a, g] + a \sum_i D_{x_i, z_i},$$

where $D_{x, z} = [R_x, R_z] + [R_x, L_z] + [L_x, L_z]$, is an inner derivation of $\mathfrak{B} = \mathfrak{A} \oplus \mathfrak{M}$, the semi-direct sum of $\mathfrak{A}$ and $\mathfrak{M}$. If the characteristic of Φ is zero, then g may be taken as zero.

PROOF. The proof is divided into four parts, and makes use of some of the results in [4].

² The referee has pointed out that since the $\sum_i D_{x_i, y_i}$ form an ideal in $\mathfrak{D}(\mathfrak{C})$, Theorem 1 is a corollary of the theorem that $\mathfrak{D}(\mathfrak{C})$ is simple. The latter result has been announced by Kaplansky in Bull. Amer. Math. Soc. vol. 60 (1954) p. 471. As far as the author knows, no proof of this result has appeared in the literature.

(1). First we may assume that $\mathfrak{M}$ is irreducible. By [4], $\mathfrak{M}$ is completely reducible. Let $\mathfrak{M} = \sum \oplus \mathfrak{M}_i$, $\mathfrak{M}_i$ invariant irreducible subspaces. If p_i is the projection of $\mathfrak{M}$ on $\mathfrak{M}_i$, then $f = \sum_i f p_i: f p_i$ is a one-cocycle of $\mathfrak{A}$ into $\mathfrak{M}_i$. Since the nucleus of $\mathfrak{M}$ is the sum of the nuclei of the $\mathfrak{M}_i$, the result follows by addition.

(2). Next we show that $\mathfrak{A}$ may be assumed to be simple. Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_r$, $\mathfrak{A}_i$ simple with identity e_i . Let $\mathfrak{M} = \sum_{i,j=0}^r \oplus \mathfrak{M}_{ij}$ be the Peirce decomposition of $\mathfrak{M}$ with respect to the $\{e_i\}$. The $\mathfrak{M}_{ij}$ are submodules, so by the above result, $\mathfrak{M} = \mathfrak{M}_{ij}$.

If $i=j=0$, there is nothing to prove. If $i=0$, then $\mathfrak{A}_k^2 = \mathfrak{A}_k$ implies $f(\mathfrak{A}_k) = 0$ for $k \neq j$. The restriction of f to $\mathfrak{A}_j$ is a one-cocycle of $\mathfrak{A}_j$ into $\mathfrak{M}$. $D_{x_j, z}$ and $R_\theta - L_\theta$ are zero when acting on $\mathfrak{A}_k$ for $k \neq j$, $x_j \in \mathfrak{A}_j$, $z, g \in \mathfrak{M} = \mathfrak{M}_{0j}$, so we may assume $\mathfrak{A} = \mathfrak{A}_j$. Similarly if $j=0$, we may take $\mathfrak{A} = \mathfrak{A}_i$.

If $\mathfrak{M} = \mathfrak{M}_{ii}$ for $i \neq 0$, then $\mathfrak{A}_k^2 = \mathfrak{A}_k$ implies $f(\mathfrak{A}_k) = 0$ for $k \neq i$. Since $D_{x_i, z}$ and $R_\theta - L_\theta$ are zero on $\mathfrak{A}_k$ for $k \neq i$, $x_i \in \mathfrak{A}_i$, $z, g \in \mathfrak{M} = \mathfrak{M}_{ii}$, we may assume $\mathfrak{A} = \mathfrak{A}_i$.

Finally, there remains the case $\mathfrak{M} = \mathfrak{M}_{ij}$, $i \neq j$, $i \neq 0$, $j \neq 0$. This is an associative bimodule for $\mathfrak{A}$. In this case, $D_{x, z}$ and $R_\theta - L_\theta$ are zero on $\mathfrak{A}_k$ for $k \neq i, j$, $x \in \mathfrak{A}_i \oplus \mathfrak{A}_j$, $z, g \in \mathfrak{M} = \mathfrak{M}_{ij}$. So we may assume $\mathfrak{A} = \mathfrak{A}_i \oplus \mathfrak{A}_j$. But since $\mathfrak{M}_{ij}$ is associative, $\mathfrak{A}_i$ and $\mathfrak{A}_j$ cannot be Cayley algebras which are simple but nonassociative. Hence $\mathfrak{A} = \mathfrak{A}_i \oplus \mathfrak{A}_j$ is associative, and the result is well-known if we set $x_i = z_i = 0$ in (3), (see [1]). Hence we may assume $\mathfrak{A}$ simple, and $\mathfrak{M} = \mathfrak{M}_{11}$, $\mathfrak{M}_{10}$ or $\mathfrak{M}_{01}$ in the Peirce decomposition relative to the identity of $\mathfrak{A}$.

(3). The case $\mathfrak{M} = \mathfrak{M}_{01}$ (or $\mathfrak{M}_{10}$).

It is easily seen [7; 8], that the associator $(1, a, m) = 0$ for $a \in \mathfrak{A}$, $m \in \mathfrak{M}$. Hence if $\mathfrak{M} = \mathfrak{M}_{01}$, then

$$\begin{aligned}(a, b, m) &= (ab)m - a(bm) \\ &= ((ab)1)m - a((b1)m) \\ &= (ab)(1m) - a(b(1m)) = 0.\end{aligned}$$

Hence $\mathfrak{M}$ is associative, $\mathfrak{A}$ must be associative, and the result is known by Hochschild's theorem. Similarly for $\mathfrak{M} = \mathfrak{M}_{10}$.

(4). The remaining case is $\mathfrak{M}$ a unital irreducible bimodule for the simple alternative algebra $\mathfrak{A}$. We extend the base field Φ to Γ so that $\mathfrak{A}_\Gamma$ is a split algebra, i.e., either Γ_n , the algebra of $n \times n$ matrices over Γ , or a Cayley algebra over Γ . Also extend $\mathfrak{M}$ to $\mathfrak{M}_\Gamma$, an $\mathfrak{A}_\Gamma$ -bimodule.

(a) If $\mathfrak{A}_\Gamma = \Gamma_n$, for $n \geq 3$, then the bimodule $\mathfrak{M}_\Gamma$ is associative, [4]. Then $\mathfrak{M}$ is associative for $\mathfrak{A}$, and the result is known.

(b) If $\mathfrak{A}_\Gamma$ is a Cayley algebra over Γ , then it is easily seen, [2], that $\mathfrak{A}$ is a Cayley algebra over Φ . Since the only unital irreducible

alternative bimodule for $\mathfrak{A} = \mathfrak{C}$ is $\mathfrak{C}$ itself, [4], f is a derivation in $\mathfrak{C}$. The result then follows from Theorem 1, taking $g = 0$.

(c) If $\mathfrak{A}_\Gamma = \Gamma_2$, and $\mathfrak{M}_\Gamma$ is an associative Γ_2 -bimodule, then $\mathfrak{M}$ is an associative $\mathfrak{A}$ -bimodule, and the result follows as before. By [4], the only other possibility is that $\mathfrak{A}_\Gamma = \Gamma_2$, and $\mathfrak{M}_\Gamma$ is a Cayley bimodule for $\mathfrak{A}_\Gamma$. In this case, the mappings $a \rightarrow R_a$, $a \rightarrow L_a$ are respectively an anti-homomorphism and a homomorphism of $\mathfrak{A}$ into the algebra of linear transformations of $\mathfrak{M}$. Thus $\mathfrak{M}$ is a Cayley bimodule for $\mathfrak{A}$. We now give a separate calculation for this last case.

Let $e_{11}, e_{12}, e_{21}, e_{22}$ be the matrix units of Φ_2 , $\mathfrak{M}$ the vector space of two dimensions acted on by Φ_2 . $\mathfrak{M}$ has a basis $v_1 = (1, 0)$, $v_2 = (0, 1)$. Let $a \rightarrow U_a$ be the representation of Φ_2 on $\mathfrak{M}$. Then $\mathfrak{M}$ is a Cayley bimodule for Φ_2 under (R_a, L_a) , where $R_a = U_{\bar{a}}$, $L_a = U_a$, where $\bar{a} = (\text{trace } a)1 - a$ is an involution in Φ_2 . $\mathfrak{M}$ is a unital alternative Φ_2 -bimodule, and $a \rightarrow L_a$, $a \rightarrow R_a$ are a homomorphism and an antihomomorphism respectively. Let $\mathfrak{D}$ be the vector space of all derivations (one-cocycles) of Φ_2 into $\mathfrak{M}$. Any element of $\mathfrak{D}$ is uniquely determined by its effect on the generators e_{12}, e_{21} of Φ_2 , and hence $\mathfrak{D}$ is four-dimensional over Φ . We exhibit a basis D_1, D_2, D_3, D_4 as follows:

	e_{12}	e_{21}
D_1	v_1	0
D_2	v_2	0
D_3	0	v_1
D_4	0	v_2

If $a \in \Phi_2$, $v \in \mathfrak{M}$, then $D_{a,v} = [R_a, R_v] + [R_a, L_v] + [L_a, L_v]$ is a derivation of the semi-direct sum $\Phi_2 \oplus \mathfrak{M}$, mapping Φ_2 into $\mathfrak{M}$, and $\mathfrak{M}$ into $\{0\}$. Its restriction to Φ_2 may be considered as an element of $\mathfrak{D}$. We show that these $D_{a,v}$ span $\mathfrak{D}$, in which case we have the desired form (1) for $f \in \mathfrak{D}$, with $g = 0$. The calculation is similar to the one in §2.

	e_{12}	e_{21}	
D_{e_{12}, v_1}	0	v_1	$D_{e_{12}, v_1} = D_3$
D_{e_{12}, v_2}	0	$3v_2$	$D_{e_{12}, v_2} = 3D_4$
D_{e_{21}, v_1}	$3v_1$	0	$D_{e_{21}, v_1} = 3D_1$
D_{e_{21}, v_2}	v_2	0	$D_{e_{21}, v_2} = D_2$

This completes the proof, except for the last statement.

If characteristic $\Phi=0$, then to show $g=0$, it again suffices to take $\mathfrak{M}$ irreducible, $\mathfrak{A}$ simple. If $\mathfrak{A}$ is a Cayley algebra or if $\mathfrak{A}$ is Φ_2 , $\mathfrak{M}$ a Cayley bimodule, then we have seen that $g=0$. If $\mathfrak{A}=\Phi_n$, $n \geq 2$, $\mathfrak{M}$ associative, then we may take $\mathfrak{M}=\Phi_n$, and the representation as the biregular one. Then $f(a)=a(R_g-L_g)$, for $g \in \Phi_n$. However, $\mathfrak{A}=\mathbb{C} \oplus \mathfrak{A}^1$, $\mathbb{C}$ the center of $\mathfrak{A}$, $\mathfrak{A}^1$ the space spanned by the commutators $[a, b] = ab - ba$ in $\mathfrak{A}$. Let $g=c + \sum_i [a_i, b_i]$. Then $R_g-L_g = \sum_i D_{a_i, b_i}$, and this proves the last statement of the theorem.

COROLLARY. *If $\mathfrak{A}$ is a separable subalgebra of an alternative algebra $\mathfrak{B}$ of characteristic not two or three, then any derivation of $\mathfrak{A}$ into $\mathfrak{B}$ can be extended to an inner derivation of $\mathfrak{B}$ which has the form $R_g-L_g + \sum D_{x_i, y_i}$, for g in the nucleus of $\mathfrak{B}$, $x_i, y_i \in \mathfrak{B}$. If characteristic $\Phi=0$, then g may be taken as zero.*

REMARKS. We remark that for characteristic zero, our theorem is Theorems 3 and 4 of [6].

One can use Theorem 2 to show that if $\mathfrak{A}$ is an alternative algebra of characteristic not two or three, $\mathfrak{N}$ its radical, $\mathfrak{A}/\mathfrak{N}$ separable, and $\mathfrak{S}_1, \mathfrak{S}_2$ two separable subalgebras of $\mathfrak{A}$ isomorphic with $\mathfrak{A}/\mathfrak{N}$, then there exists a nonsingular linear transformation $G=I+D$, D in the radical of the enveloping associative algebra of $\mathfrak{A}$, such that $\mathfrak{S}_1^G = \mathfrak{S}_2$.

Finally we note that our examples showed that Theorem 2 is false for characteristic three. In this case, commutation by any element of $\mathfrak{M}$ is a one-cocycle. However, even if one removes the restriction that g be in the nucleus of $\mathfrak{M}$, the Cayley algebra will still provide a counter-example to Theorem 2 for characteristic three.

REFERENCES

1. G. Hochschild, *The cohomology groups of an associative algebra*, Ann. of Math. (2) vol. 46 (1945).
2. N. Jacobson, *Cayley numbers and normal simple Lie algebras*, Duke Math. J. vol. 5 (1939).
3. ———, *General representation theory of Jordan algebras*, Trans. Amer. Math. Soc. vol. 70 (1951).
4. ———, *Structure of alternative and Jordan bimodules*, Osaka Math. J. vol. 6 (1954).
5. R. D. Schafer, *Inner derivations of nonassociative algebras*, Bull. Amer. Math. Soc. vol. 55 (1949).
6. ———, *Representations of alternative algebras*, Trans. Amer. Math. Soc. vol. 72 (1952).
7. M. Zorn, *Theorie der alternativen Ringe*, Abh. Math. Sem. Hamburgischen Univ. vol. 8 (1930).
8. ———, *Alternative rings and related questions I: Existence of the radical*, Ann. of Math. (2) vol. 42 (1941).