NORMAL SUBGROUPS OF MONOMIAL GROUPS

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1. Introduction. Let U be a set, $o(X) = B = \aleph_u$, $u \geq 0$, where $o(U)$ means the number of elements of U . Let H be a fixed group. A monomial substitution y is a transformation that maps every x of U in a one-to-one fashion into an x of U multiplied by an element h_x of H . Multiplication of substitutions means successive applications. The set of all monomial substitutions forms a group Σ . Ore [1] has studied this group for finite U , and some of his results have been generalized to general U in [3].

The normal subgroups for one subgroup of Σ have been determined [3]. This paper extends those results to the extent of determining the structure of all normal subgroups of various subgroups of Σ in a rather general case. These results are stated in Theorems 1, 2, 3, 4, 5, 6.

2. Definitions. Let d be the cardinal of the integers. Let B be an infinite cardinal; B^+ , the successor of B ; U , a set such that $o(U) = B$; and C such that $d < C \leq B^+$. Let H be a fixed group and e the identity of H . Denote by $\Sigma = \Sigma(H, B, d, C)$ the *monomial group* of U over H whose elements are of the form

$$(1) \quad y = \begin{pmatrix} \cdots, & x_\epsilon, & \cdots \\ \cdots, & h_\epsilon x_{i_\epsilon}, & \cdots \end{pmatrix}$$

where only a finite number of the h_ϵ are not e and the number of x not mapped into themselves is less than C . Any element of Σ may be written in the form

$$y = \begin{pmatrix} \cdots, & x_\epsilon, & \cdots \\ \cdots, & h_\epsilon x_\epsilon, & \cdots \end{pmatrix} \begin{pmatrix} \cdots, & x_\epsilon, & \cdots \\ \cdots, & ex_{i_\epsilon}, & \cdots \end{pmatrix}$$

or $y = vs$ where v sends every x into itself, every h of s is e . Elements of the form of

$$v = \begin{pmatrix} \cdots, & x_\epsilon, & \cdots \\ \cdots, & h_\epsilon x_\epsilon, & \cdots \end{pmatrix} = \{ \cdots, h_\epsilon, \cdots \}$$

are *multiplications* and all such elements form a normal subgroup,

Presented to the Society, December 28, 1956; received by the editors March 18, 1956 and, in revised form, December 22, 1956.

the basis group $V(B, d) = V$ of Σ . The h_e of y are called the *factors* of y . Elements of the form of s are *permutations* and all such elements form a subgroup, the *permutation group*, $S(B, C) = S$ of $\Sigma(H; B, d, C)$. Cycles of s will also be written as $(x_1, \dots, x_n)$ and $(\dots, x_{-1}, x_0, x_1, \dots)$. Baer [2] has shown that the normal subgroups of $S(B, C)$ are the alternating group, $A = A(B, d)$, and $S(B, D)$ where $d \leq D \leq C$. Let E be the identity of Σ , I the identity of S .

3. Normal subgroups of Σ contained in the basis group.

THEOREM 1. *Let $G_1 \subset G$ be normal subgroups of H such that G/G_1 belongs to the center of H/G_1 . Let N be the set of multiplications with a finite number of different from e factors subject to the conditions that the factors are in G and their product, in any order, belongs to G_1 . N is a normal subgroup of Σ contained in the basis group. Conversely, let N be a normal subgroup of Σ contained in the basis group. For every v of N form all possible products of its nonidentity factors. This set G_1 is a normal subgroup of H . The set G consisting of every element of h that occurs as a factor in any v of N is also a normal subgroup of H . Further, G/G_1 belongs to the center of H/G_1 .*

PROOF. It is clear that N is a subgroup. Let $y_1 = v_1 s_1$ be any element of Σ . Then $y_1 v y_1^{-1} = v_1 s_1 v s_1^{-1} v_1^{-1} = v_1 v_2 v_1^{-1}$ where v_2 belongs to N . Since G/G_1 is in the center of H/G_1 the product of the factors of $v_1 v_2 v_1^{-1}$ is in G_1 and N is normal in Σ .

If N is normal in Σ then an argument similar to that in [3, pp. 204–207] shows it has the structure described in the theorem.

4. Normal subgroups of Σ not contained in the basis group. The problem of finding all normal subgroups of Σ will be concluded by finding those normal subgroups not in V .

LEMMA 1. *Let M be normal in Σ , $V \ntriangleleft M$. Then $N = M \cap V$ is normal in Σ and $G = H$, i.e., the factors in any fixed position run through H and H/G_1 is Abelian.*

PROOF. Choose $y = v s \in M$ such that $s \neq I$. Let $v = \{k_1, k_2, \dots\}$ be arbitrary in V . Then the multiplication $y^{-1} v^{-1} y v$ is in M . Let s send x_ϵ into x_{i_ϵ} with $\epsilon \neq i_\epsilon$. The factor occurring in the position occupied by x_{i_ϵ} is $h_\epsilon^{-1} k_\epsilon^{-1} h_\epsilon k_{i_\epsilon}$. Since v is arbitrary and $i_\epsilon \neq \epsilon$ we may choose the factors $k_\epsilon^{-1}, k_{i_\epsilon}$ in such a way that the factor above is arbitrary in H .

LEMMA 2. *Let M be normal in Σ , $V \ntriangleleft M$. Then $P = M \cap S$ is normal in S and $P \neq E$.*

PROOF. Let y be an element of M and $y = v s$ where $s \neq I$. Since y

has only a finite number of different from e factors, M must contain a y' conjugate to y where the finite cycles of y have been written in normal form, [1, p. 20]. If y contains an infinite cycle then y' contains an infinite cycle in the form,

$$\left(\begin{array}{ccccccc} \cdots, & x_{-1}, & x_0, & x_1, & \cdots \\ \cdots, & x_0, & gx_1, & x_2, & \cdots \end{array} \right).$$

If y' contains a finite cycle then [1, pp. 35-36] M contains a permutation. If y' contains only infinite cycles let $s_1 = (2, 3)$. Then the commutator

$$\begin{aligned} (y')^{-1}s_1y's_1^{-1} \\ = \left(\begin{array}{cccccccc} \cdots, & x_{-1}, & x_0, & & x_1, & x_2, & x_3, & x_4, & x_5, & \cdots \\ \cdots, & x_{-1}, & x_0, & g^{-1}gx_1, & x_3, & x_4, & x_2, & x_5, & \cdots \end{array} \right) = (2, 3, 4) \end{aligned}$$

belongs to M .

THEOREM 2. *If $M = N \cup P$, where N is as in Lemma 1 and P is a normal subgroup of S , then M is normal in Σ .*

PROOF. Let $y = vs$ be any element of M . Let $y_1 = v_1s_1$ be any element of Σ . Then $y_1yy_1^{-1} = y_1vy_1^{-1}v_1s_1s_1^{-1}v_1^{-1} = v_2v_1s_2v_1^{-1}$ where $v_2 \in N$, $s_2 \in P$ since N is normal in Σ and P is normal in S . It is sufficient to show $v_1s_2v_1^{-1}$ is in M . A computation shows

$$\begin{aligned} v_1s_2v_1^{-1} &= \{h_1, \cdots, h_\epsilon, \cdots\} \left(\begin{array}{cccc} x_1, & \cdots, & x_\epsilon, & \cdots \\ x_{i_1}, & \cdots, & x_{i_\epsilon}, & \cdots \end{array} \right) \{h_1^{-1}, \cdots, h_\epsilon^{-1}, \cdots\} \\ &= \{h_1h_{i_1}^{-1}, \cdots, h_\epsilon h_{i_\epsilon}^{-1}, \cdots\} = s_2 = v_3s_2 \end{aligned}$$

where all but a finite number of the factors of v_3 are e . Since H/G_1 is Abelian the product of the factors of v_3 is in G_1 . Therefore, $v_3 \in N$, $s_2 \in P$ and M is normal in Σ .

THEOREM 3. *If $V \ntriangleleft M$, M normal in Σ , $M \cap S = P = S(B, D)$, $d \leq D \leq C$, $M \cap V = N$ then $M = N \cup P$.*

PROOF. Assume there exists $y \in M$, $y = vs$, and $s \notin P$. This means s moves D or more of the elements of U . Then y is conjugate to y' of M where y' in its cyclic normal form has d -cycles written in the form

$$\left(\begin{array}{cccc} \cdots, & x_{-1}, & x_0, & x_1, & \cdots \\ \cdots, & x_0, & gx_1, & x_2, & \cdots \end{array} \right).$$

Construct s_1 as follows. For each n -cycle, $n \geq 3$, of y' of the form

$$c = \begin{pmatrix} x_1, \dots, x_n \\ x_2, \dots, ax_1 \end{pmatrix}$$

let s_1 have the cycle (x_1, x_2) . For each pair of 2-cycles

$$\begin{pmatrix} x_1, & x_2 \\ x_2, & bx_1 \end{pmatrix}, \quad \begin{pmatrix} x_3, & x_4 \\ x_4, & cx_3 \end{pmatrix}$$

of y' let s_1 have the cycle $(x_1, x_3)(x_2, x_4)$. If there is a 2-cycle

$$\begin{pmatrix} x_\alpha, & x_\beta \\ x_\beta, & dx_\alpha \end{pmatrix}$$

of y' left over let s_1 send s_α, s_β into themselves. For each d -cycle of y'

$$\begin{pmatrix} \dots, & x_{-1}, & x_0, & x_1, & \dots \\ \dots, & x_0, & gx_1, & x_2, & \dots \end{pmatrix}$$

let s_1 have the cycles

$$\begin{pmatrix} x_0 \\ x_0 \end{pmatrix} \begin{pmatrix} x_n, & x_{-n} \\ x_{-n}, & x_n \end{pmatrix} \quad \text{for } n = 1, 2, \dots$$

Form the commutator $(y')^{-1}s_1y'(s_1)^{-1}=y_1$ which is in M . For each n -cycle, $n \geq 3$, of y' , y_1 contains (x_1, x_2, x_3) . For each pair of 2-cycles y_1 contains $(x_1, x_3)(x_2, x_4)$. For each d -cycle of y' , y_1 will contain

$$\begin{pmatrix} \dots, & x_{-3}, & x_{-2}, & x_{-1}, & x_0, & & x_1, & x_2, & x_3, & \dots \\ \dots, & x_{-5}, & x_{-4}, & x_{-3}, & x_{-2}, & g^{-1}gx_{-1}, & x_0, & x_1, & \dots \end{pmatrix} \\ = \begin{pmatrix} \dots, & x_2, & x_0, & & x_{-2}, & \dots \\ \dots, & x_0, & x_{-2}, & x_{-4}, & \dots \end{pmatrix} \begin{pmatrix} \dots, & x_3, & x_1, & x_{-1}, & \dots \\ \dots, & x_1, & x_{-1}, & x_{-3}, & \dots \end{pmatrix}.$$

This shows that M contains a permutation which moves the same number of x 's as the s of $y=vs$, contradicting $P=S(B, D)$.

If $y=vs \in M$ then since $s \in M$, s^{-1} also belongs to M , and $ys^{-1}=v \in M$.

THEOREM 4. *Let $M \subset \Sigma$, M normal in Σ , $M \cap S = A(B, d)$, $M \cap V = N$ and $M/N \cong A(B, d)$. Then $M = N \cup A(B, d)$.*

PROOF. Since $M \supset (N \cup A)$ and $N \cup A$ is normal in Σ it follows that $N \cup A$ is normal in M . Thus $N \cup A/N$ is normal in M/N which is simple since $M/N \cong A(B, d)$. Therefore, $N \cup A = M$.

THEOREM 5. *Let $M \subset \Sigma$, M normal in Σ , $V \ntriangleleft M$, $M \cap S = A$, $M \cap V = N$, and M/N not $\cong A$. Then $M = N \cup A \cup L$, where L is the cyclic subgroup generated by*

$$y = \begin{pmatrix} x_1 & x_2 \\ x_2 & ax_1 \end{pmatrix}$$

with a^2 in G_1 . If L_1 is the cyclic subgroup generated by

$$y_1 = \begin{pmatrix} x_1 & x_2 \\ x_2 & bx_1 \end{pmatrix},$$

where $b^2 \in G_1$, $M_1 = N \cup A \cup L_1$ is M if, and only if, a and b are in the same coset of G_1 .

PROOF. M contains an element $y = vs$ with s in $S(B, C)$, $C \geq d$, and $s \notin A$. Otherwise, for every $y = vs$ of M , $s \in A$. Then every element of $V \cup M$ would be of the form $y = v_1 v s = v_2 s$ and $(V \cup M)/V \cong A$. But $(V \cup M)/V \cong M/N$ not $\cong A$. Now if $y = vs$ with $s \in S(B, C)$ with $C > d$ the method used to prove Theorem 3 will lead to $M \cap S \neq A$, a contradiction. So assume $s \in S(B, d)$. The product of any 2 elements of M outside $N \cup A$ is in $N \cup A$ since the permutation component is finite and even. Let x_1, x_2 be two elements s leaves fixed. The permutations $s^{-1}(x_1, x_2)$ belong to $A = P \subset M$. Therefore, $ys^{-1}(x_1, x_2) = vss^{-1}(x_1, x_2)$ belongs to M . There is an element v_1 in N such that $v_1 v(x_1, x_2)$ can be reduced to the form

$$y = \begin{pmatrix} x_1 & x_2 \\ x_2 & ax_1 \end{pmatrix}$$

because the factors of elements of N are unrestricted except that the product is in G_1 . This element squared is in $N \subset M$ so $a^2 \in G_1$.

THEOREM 6. Let $M = N \cup A \cup L$, where N is as in Lemma 1 and L is the cyclic group generated by

$$y = \begin{pmatrix} x_1 & x_2 \\ x_2 & ax_1 \end{pmatrix}$$

with $a^2 \in G_1$. Then M is normal in Σ .

PROOF. It is sufficient to show $vsy(vs)^{-1}$ belongs to M for all vs of Σ because y^2 belongs to N . This may be reduced to showing sys^{-1} and vyv^{-1} belong to M . Let s be arbitrary in S and

$$s = \begin{pmatrix} x_i, \dots, x_j, \dots \\ x_1, \dots, x_2, \dots \end{pmatrix}.$$

Then

$$sys^{-1} = \begin{pmatrix} x_i & x_j \\ x_j & ax_i \end{pmatrix}.$$

But there exists an s_1 of A such that

$$s_1 = \begin{pmatrix} x_i, \dots, x_j, \dots \\ x_1, \dots, x_2, \dots \end{pmatrix}.$$

But $sys^{-1} = s_1ys_1^{-1}$ belongs to $(A \cup L \cup A) \subset M$. Let v be arbitrary in V and $v = \{h_1, \dots, h_e, \dots\}$. The commutator $y^{-1}vyv^{-1} = \{a^{-1}h_2ah_1^{-1}, h_1h_2^{-1}, e, \dots\}$ belongs to $N \subset M$ if $a^{-1}h_2ah_1^{-1}h_1h_2^{-1}$, belongs to G_1 . Since H/G_1 is Abelian the desired result follows.

THEOREM 7. *If $C = \mathfrak{N}_n$, where n is finite, and if H is finite then the number of normal subgroups of Σ is finite.*

PROOF. This follows from the fact that there are only a finite number of choices of (1) normal subgroups of $S(B, C)$, (2) normal subgroups of H , (3) cosets of G_1 .

BIBLIOGRAPHY

1. O. Ore, *Theory of monomial groups*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 15-64.
2. R. Baer, *Die Kompositionsreihe der Gruppe aller eineindeutigen Abbildungen einer unendlichen Reihe auf sich*, Studia Math. vol. 5 (1934) pp. 15-17.
3. R. Crouch, *Monomial groups*, Trans. Amer. Math. Soc. vol. 80 (1955) pp. 187-215.

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